Complex Represenation of Waves

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Cornell University

Department of Physics

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1 Introduction

The previous set of notes "Reflection and Transmission at a Change in Medium," determined the form of *scattered*, either transmitted and reflected, pulses at a change in medium by working with the general solution to the wave equation. As we shall see below, for the special case of incoming traveling waves of sinusoidal form, the form of transmission and reflection may be determined directly by an appeal to general physical principles. Because all incoming pulse shapes may ultimately be decomposed into a sum of sinusoidal functions by a mathematical procedure called Fourier analysis, by studying scattering of sinusoidal functions one can in fact determine the form of scattering for any pulse shape. Although we shall not study Fourier analysis in this course, it is still useful for us to study scattering amplitudes for sinusoidal pulses at a change in medium because, as we have seen, reflection and transmission amplitudes are independent of the incoming pulse shape. Thus, for any shape pulse, the transmission and reflection scattering amplitudes will be exactly what we compute for sinusoidal pulses.

We also study scattering of sinusoidal pulses at a change in medium because this problem reveals far more general ideas which may be applied to find the solution to the equations of motion for any system driven by an incoming wave of fixed frequency, such as the interference and diffraction setups which we shall study in the next set of notes, “Interference and Diffraction.”

2 Complex representation for traveling waves

To introduce the use of the complex representation for waves, we first consider a sinusoidal pulse of the form

$$f(u) = A\cos(ku+\phi_0) = \text{Re}\,\left(\underline{A}e^{iku}\right), \tag{1}$$

where $\underline{A} \equiv Ae^{i\phi_0}$ is a complex amplitude. Note that (1) is in exactly analogous form to what we did previously in the time domain for simple harmonic motion. The only difference is that now we are in the space domain, so that instead of angular frequency ω and time t, we use wave vector k and position u.

Next, we consider the form of a traveling wave solution made from such a pulse,

$$\begin{aligned} s(x,t) &= f(x-ct) \\ &= \text{Re}\,\left(\underline{A}e^{ik(x-ct)}\right) \\ &= \text{Re}\,\left(\underline{A}e^{ikx}e^{-ickt}\right) \\ &= \text{Re}\,\left(\underline{A}e^{ikx}e^{-i\omega t}\right) \\ &= \text{Re}\,\left(e^{-i\omega t}\underline{A}e^{ikx}\right) \\ &= \text{Re}\,\left(e^{-i\omega t}\underline{Q}(x)\right), \end{aligned} \tag{2}$$

where we have used the dispersion relation $\omega = ck$ and defined the complex amplitude at position x to be

$$\underline{Q}(x) \equiv \underline{A}e^{ikx}. \tag{3}$$

The final line above shows that the motion at each point x in such a traveling wave is just simple harmonic motion in time at the wave frequency ω, with an amplitude and phase determined by $\underline{Q}(x)$. Specifically,

$$\begin{aligned} s(x,t) &= \text{Re}\,\left(e^{-i\omega t}\underline{Q}(x)\right) \\ &= \text{Re}\,\left(e^{-i\omega t}\left|\underline{Q}(x)\right|e^{i\phi_{\underline{Q}(x)}}\right) \\ &= \text{Re}\,\left(\left|\underline{Q}(x)\right|e^{-i(\omega t-\phi_{\underline{Q}(x)})}\right) \\ &= \left|\underline{Q}(x)\right|\cos\left(\omega t-\phi_{\underline{Q}(x)}\right). \end{aligned}$$

Thus, $\underline{Q}(x)$ determines completely the motion of point x as simple harmonic motion with amplitude $\left|\underline{Q}(x)\right|$ and phase $-\phi_{\underline{Q}(x)}$. In the next section, we will find that the solution for wave propagation problems with input waves at a fixed frequency ω always has the form (2), so that determining the solution for the motion of the system boils down just to finding the complex wave amplitude function $\underline{Q}(x)$.

An important way of understanding the complex amplitude function $\underline{Q}(x)$ is to consider changes in the motion of a wave wave relative to its motion at a reference point a. To do this, we rewrite (2) as

$$s(x,t) = \text{Re}\,\left(\left[e^{-i\omega t}\underline{A}e^{ika}\right]e^{ik(x-a)}\right), \tag{4}$$

where the term in square brackets represents what we would have for the motion at the point a, and the factor $e^{ik(x-a)}$ represents the change in the motion in going from the point a to the point x. Because the motion at all points is simple harmonic, the only possible differences in the motion as we go from point to point are in the amplitude or in the phase of the motion.

In the plane wave motion we consider here, a traveling wave moves along as a fixed shape, and so we expect the amplitude of the motion, (max − min)/2, to be constant for each point in space. (As waves spread outward from a point, as in the next set of notes, we in general can expect a decay in amplitude with distance.) The constantness of the amplitude corresponds to the fact that the propagation factor $e^{ik(x-a)}$ has amplitude $\left|e^{ik(x-a)}\right| = 1$. Next, we generally do expect there to be a change in phase when moving from the point a to the point x because there is a time delay for maxima passing a to reach x. The time delay for traveling the distance $L = x - a$ is L/c. To convert this to a phase in radians, we measure the delay in periods and multiply by 2π: $\Delta\phi = 2\pi(L/c)/T = (\omega/c)L = kL$, precisely the phase appearing in the phase factor $e^{ik(x-a)}$. Thus, the propagation factor $e^{ik(x-a)}$ corresponds precisely to the phase delay as wave peaks pass a propagating along to x. Note that the above argument is unaffected by whether the distance L is traveled from left to right or from right to left. We thus have the following rule,

> **Propagation factor:**
> The effect of propagating a wave a distance L (measured as a positive value whether the wave moves to the right or left) appears in the complex representation as multiplication by the complex phase factor e^{ikL}. For plane waves and waves moving in one dimension, this is the only factor. If the wave spreads out from a point in two or three dimensions, there in general may also be an amplitude decay factor.

3 Solutions for waves at an interface

The previous set of notes "Reflection and Transmission at a Change in Medium," gives the general solution for waves at a change in medium from Region 0 ($x < 0$) to Region 1 ($x > 0$) as

$$\begin{aligned} s_0(x \le 0, t) &= t_0(x - c_0 t) + R_{0\to1} t_0\left(-(x + c_0 t)\right) + T_{1\to0} t_1\left(\frac{c_1}{c_0}(x + c_0 t)\right) && (5) \\ s_1(x \ge 0, t) &= t_1(x + c_1 t) + R_{1\to0} t_1\left(-(x - c_1 t)\right) + T_{0\to1} t_0\left(\frac{c_0}{c_1}(x - c_1 t)\right). \end{aligned}$$

Our strategy is to begin with this form, investigate what it looks like in the complex representation, and then see how we could immediately write down the result in the complex representation.

Sending a sinusoidal pulse in from Region 0 ($x < 0$) toward Region 1 corresponds to the case

$$\begin{aligned} t_0(u) &= \text{Re}\left(\underline{A} e^{ik_0 u}\right) && (6) \\ t_1(v) &= 0, && (7) \end{aligned}$$

where we use the subscript on k_0 to be sure to label it as the wave vector in Region 0.

3.1 Region 0

According to (5), the solution at any point x_0 in Region 0 thus has the form

$$\begin{aligned} s_0(x_0 \le 0, t) &= t_0(x_0 - c_0 t) + R_{0\to1} t_0(-(x_0 + c_0 t)) \\ &= \text{Re}\left(\underline{A} e^{ik_0(x_0 - c_0 t)}\right) + R_{0\to1}\text{Re}\left(\underline{A} e^{ik_0(-(x_0 + c_0 t))}\right) \\ &= \text{Re}\left(\underline{A} e^{ik_0 x_0} e^{-ik_0 c_0 t} + R_{0\to1}\underline{A} e^{-ik_0 x_0} e^{-ik_0 c_0 t}\right) \\ &= \text{Re}\left(e^{-i\omega t}\underline{A}\left(e^{ik_0 x_0} + R_{0\to1} e^{-ik_0 x_0}\right)\right) \\ &= \text{Re}\left(e^{-i\omega t}\underline{Q}_0(x_0)\right), && (8) \end{aligned}$$

where the wave frequency is $\omega = k_0 c_0$, and we define a complex amplitude for the wave solution in Region 0 to be

$$\underline{Q}_0(x_0 \le 0) \equiv \underline{A}\left(e^{ik_0 x_0} + R_{0\to1} e^{-ik_0 x_0}\right). \quad (9)$$

Conveniently, we find the total solution in Region 0 to be of the form (2) of simple harmonic motion at the incoming wave frequency. This is just the general consequence of the fact that the response of a system driven at frequency ω is motion at that same frequency. Given this knowledge, the complex amplitude $\underline{Q}_0(x_0)$ then determines entirely the motion for Region 0.

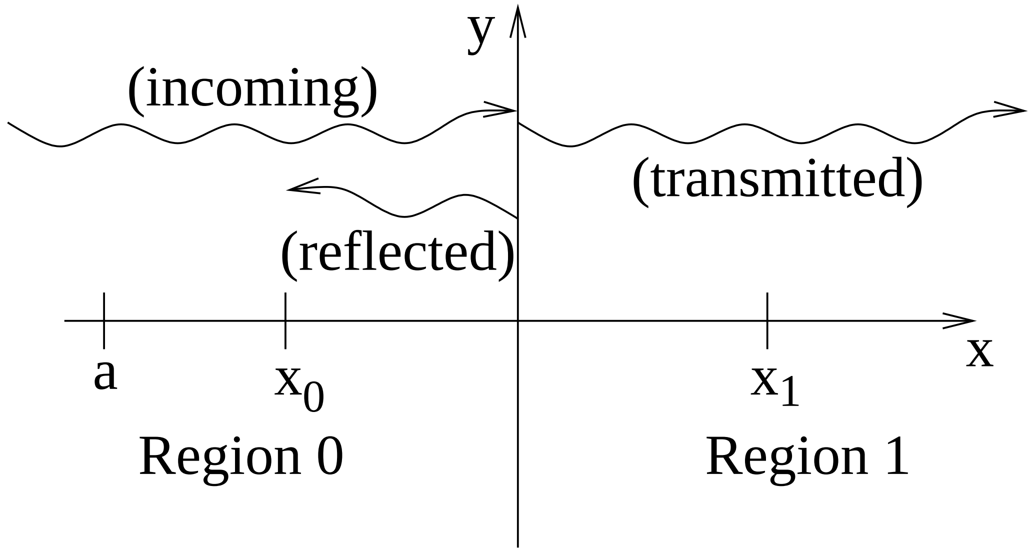

Figure 1: Scattering (reflection and transmission) from a change in medium from Region 0 ($x < 0$) to Region 1 ($x > 0$).

To write this result down directly using the complex representation, we consider the motion at each point x_0 relative to the motion associated with the incoming wave at point a, which from our previous discussion we know to be Re $\left(e^{-i\omega t}\underline{A}e^{ik_0 a}\right)$ for some complex amplitude $\underline{A}$. We expect two waves contribute to the motion at each point $x_0 < 0$, the incoming wave and the reflected wave. (See Figure 1.) By the principle of superposition, the final motion will be the sum each of these.

The motion due to the incoming wave at point x_0, a distance $L = (x_0 - a)$ from the point a, will include an additional propagation factor of $e^{ik_0(x-0-a)}$. The first contribution to the motion at x_0 is thus

$$\text{Re}\left(e^{-i\omega t}\underline{A}e^{ik_0 a} \cdot e^{ik_0(x_0-a)}\right).$$

The second contribution comes from the reflected wave. To compute this contribution at point x_0, we make a number of relative comparisons. First, we compare the motion of the incoming wave at a to the motion of the same wave at the interface, $x = 0$. The distance propagated is now $L = (0 - a) = -a$, and so we must include a factor of $e^{ik_0(-a)}$ to find the motion of the incoming wave at $x = 0$. Next, we know that the motion of the reflected wave at the interface matches the motion of the incoming wave at the interface, except for the reflection amplitude factor $R_{0\to1}$. Thus, the motion of the reflected wave at $x = 0$ is just the motion of the reference point times two correction factors, $e^{ik_0(-a)}$ and $R_{0\to1}$. Finally, relative to the motion of the reflected wave at $x = 0$, the motion induced by the reflected wave at x_0, must contain one final propagation factor $e^{ik_0(-x_0)}$. Thus, the reflected wave contribution to the motion at x_0 is

$$\text{Re}\left(e^{-i\omega t}\underline{A}e^{ik_0 a} \cdot e^{ik_0(-a)} \cdot R_{0\to1} \cdot e^{ik_0(-x_0)}\right).$$

Adding our two contributions together, we find that we are able to directly write a result equivalent to (8),

$$s_0(x_0, t) = \text{Re}\left(e^{-i\omega t}\underline{A}e^{ik_0 a} \cdot \left[e^{ik_0(x_0-a)} + e^{ik_0(-a)} \cdot R_{0\to1} \cdot e^{ik_0(-x_0)}\right]\right).$$

3.2 Region 1

Eq. (5) also gives the solution in Region 1,

$$s_1(x_1 \geq 0, t) \quad = \quad T_{0\to1} t_0 \left(\frac{c_0}{c_1}(x_1 - c_1 t)\right)$$

$$
\begin{aligned}
&= T_{0\to1}\text{Re}\left(\underline{A}e^{ik_0\left(\frac{c_0}{c_1}(x_1-c_1)\right)}\right) \\
&= \text{Re}\left(\underline{A}T_{0\to1}e^{i\frac{c_0k_0}{c_1}x_1}e^{-ic_0k_0t}\right) \\
&= \text{Re}\left(\underline{A}T_{0\to1}e^{ik_1x_1}e^{-i\omega t}\right) \\
&= \text{Re}\left(e^{-i\omega t}\underline{A}T_{0\to1}e^{ik_1x_1}\right) \\
&= \text{Re}\left(e^{-i\omega t}\underline{Q}_1(x_1)\right),
\end{aligned} \tag{10}
$$

where, in the second line from the bottom, we have used the dispersion relation of Region 1 to identify $(c_0/c_1)k_0 = \omega/c_1$ as just the wave vector k_1 from Region 1. Also, in the last line we define a new complex amplitude for the wave motion in Region 1,

$$
\underline{Q}_1(x_1 \geq 0) \equiv \underline{A}T_{0\to1}e^{ik_1x_1}. \tag{11}
$$

Thus, we once again find that all points in the system respond with simple harmonic motion at the same driving frequency ω, so that knowledge of $\underline{Q}_1(x_1)$ completely determines the motion in Region 1. For instance, the amplitude of the motion for each point in Region 1 is $\left|\underline{Q}_1(x_1)\right| = \left|\underline{A}T_{0\to1}e^{ik_1x_1}\right| = |\underline{A}| \cdot |T_{0\to1}| \cdot \left|e^{ik_1x_1}\right| = |\underline{A}| \cdot |T_{0\to1}|$, just the amplitude of the incoming wave times the magnitude of the transmission amplitude.

Finally, we see that we can again write this result directly in terms of relative complex amplitudes. Relative to the motion at point $x = a$, the motion of the transmitted wave at x_1 should contain a factor of $e^{ik_0(-a)}$ to give the motion of the incoming wave at $x = 0$, a factor of $T_{0\to1}$ to give the motion of the transmitted wave at $x = 0$ relative to the motion of the incoming wave at the same point, and a factor of $e^{ik_1x_1}$ to give the motion of the traveling wave after propagating the distance $|L| = x_1 - 0$ from $x = 0$ to the point x_1. We thus can immediately write a result equivalent to (10),

$$
s_1(x_1,t) = \text{Re}\left(e^{-i\omega t}\underline{A}e^{ik_0a} \cdot \left[e^{ik_0(-a)} \cdot T_{0\to1} \cdot e^{ik_1x_1}\right]\right).
$$

Note that in the case we do not have a sum of terms because only a single wave contributes to the motion in Region 1. (See Figure 1.)

3.3 General Lesson

Generally, we can find the complex wave amplitude at any point in a problem by summing the amplitudes for all waves which contribute at a given point (principle of superposition), where we determine the complex amplitudes for these waves by comparing the motions of the different waves at different points in space, including a factor for each comparison. Each of these comparisons may be thought of as fundamental event in the history of how the wave began at the reference point and ended up at the final observation point. Such fundamental events include propagation from point a to point b, and reflection or transmission at boundaries.

This perspective allows us to summarize our general lesson as

Sum over histories:
The complex amplitude for wave motion at point x equals the complex amplitude of the incoming wave at a reference point a, times the sum of the amplitudes for each possible history h for how the incoming wave can get from a to x. The amplitude associated with each history is the product of the complex amplitudes $\underline{a}(e)$ for each fundamental event e in that history. Mathematically,

$$
Q(x) = Q(a)\sum_h\left(\prod_{e\in h}\underline{a}(e)\right). \tag{12}
$$

Here, $\underline{a}(e) = e^{ik|b-a|}$ for an event e of propagation from point a to point b, and $\underline{a}_e$ is the corresponding reflection or transmission amplitude for a scattering event.

4 Using the complex representation to find scattering amplitudes

Now that we are able to write down the forms (8) and (10) directly, it is then a relatively simple matter to compute scattering amplitudes such as $T_{0\to1}$ and $R_{0\to1}$ directly from the boundary conditions. For notational convenience, in this section we shall drop the writing of the subscripts "$0 \to 1$" and refer to these amplitudes as T and R, respectively. Eqs. (8,10) then become

$$\begin{aligned} s_0(x \le 0, t) &= \text{Re}\left(e^{-iwt}\underline{A}\left[e^{ik_0x} + Re^{-ik_0x}\right]\right) \\ s_1(x \ge 0, t) &= \text{Re}\left(e^{-iwt}\underline{A}\left[Te^{ik_1x}\right]\right) \end{aligned} \tag{13}$$

Substituting (13) into the consistency condition, $s_0(x=0,t) = s_1(x=0,t)$, we find

$$\begin{aligned} \text{Re}\left(e^{-iwt}\underline{A}\left[e^{ik_0\cdot 0} + Re^{-ik_0\cdot 0}\right]\right) &= \text{Re}\left(e^{-iwt}\underline{A}\left[Te^{ik_1\cdot 0}\right]\right) \\ \text{Re}\left(e^{-iwt}\underline{A}\left[1+R\right]\right) &= \text{Re}\left(e^{-iwt}\underline{A}\left[T\right]\right), \end{aligned}$$

so that

$$\text{Re}\left(e^{-iwt}\underline{A}\left[1+R-T\right]\right) = 0.$$

Now, Re $\left(e^{-i\omega t}\underline{Q}\right)$ can only be zero for all times t if the harmonic motion which it represents has zero amplitude, which implies $\underline{Q} = 0$. We thus conclude $1 + R - T = 0$, so that

$$1 + R = T \tag{14}$$

Next, substituting (13) into the force balance condition, $B_0 \partial s_0(x=0,t)/\partial x = B_1 \partial s_1(x=0,t)/\partial x$, we find

$$\begin{aligned} B_0\text{Re}\left(e^{-iwt}\underline{A}\left[ik_0e^{ik_0\cdot 0} - ik_0Re^{-ik_0\cdot 0}\right]\right) &= B_1\text{Re}\left(e^{-iwt}\underline{A}\left[ik_1Te^{ik_1\cdot 0}\right]\right) \\ \text{Re}\left(e^{-iwt}\underline{A}B_0\left[ik_0 - ik_0R\right]\right) &= \text{Re}\left(e^{-iwt}\underline{A}B_1\left[ik_1T\right]\right), \end{aligned}$$

where the factors of ik come down as we take the derivatives with respect to x before substituting $x = 0$. Combining terms we find

$$\text{Re}\left(e^{-iwt}\underline{A}i\left[B_0k_0(1-R) - B_1k_1T\right]\right) = 0,$$

for all times t. Following the same logic which led to (14), we find

$$B_0k_0(1-R) = B_1k_1T.$$

Using the fact that $Bk = B(\omega/c) = Z\omega$, this simplifies to

$$\begin{aligned} Z_0\omega(1-R) &= Z_1\omega T \\ Z_0(1-R) &= Z_1T. \end{aligned} \tag{15}$$

Finally, to find the reflection and transmission amplitudes, we combine (14) and (15). To find T, we take $Z_0\times$(14)+(15):

$$2Z_0 = (Z_0 + Z_1)T.$$

This gives precisely our previous result,

$$T = \frac{2Z_0}{Z_0 + Z_1}.$$

And, to find R, we take $Z_1\times$(14)$-$(15):

$$(Z_1 - Z_0) + (Z_1 + Z_0)R = 0.$$

Again, we find our previous result,

$$R = \frac{Z_0 - Z_1}{Z_0 + Z_1}.$$

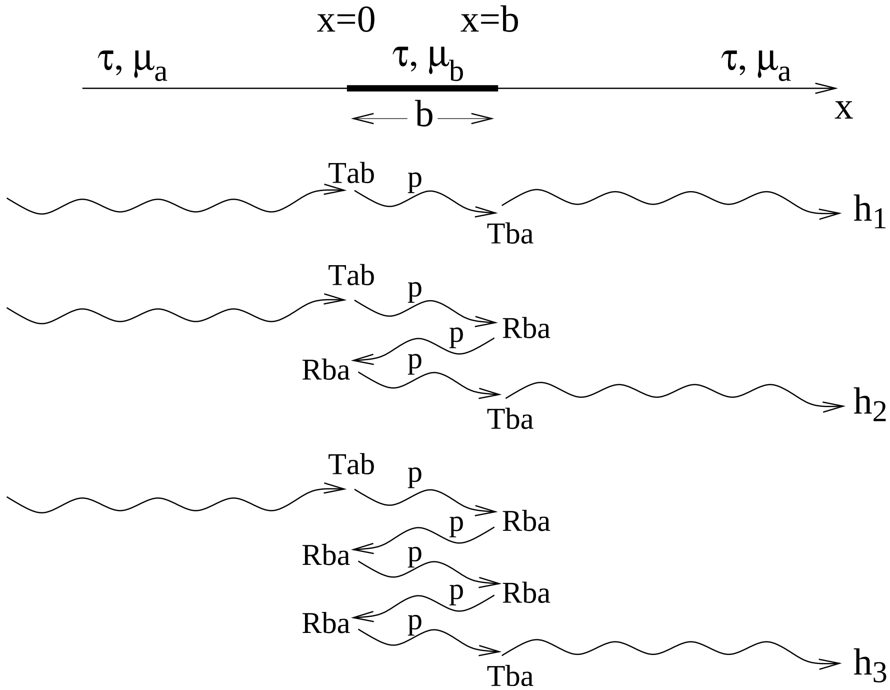

Figure 2: Transmission of waves through a barrier on a string

5 Example of sum over histories

As a more involved example of the sum over histories, consider transmission of waves of wave vector k_a on a string a of tension and mass per unit length τ and μ_a through a barrier made up of a short segment of a heavier string of mass per unit length $\mu_b > \mu_a$ and length b. (See Figure 2.) Note that because the segments of the string are in equilibrium before we allow any wave motion, the horizontal tensions τ in all segments must be equal.

The relevant quantities for propagation in this problem are the respective wave speeds, $c_a = \sqrt{\tau/\mu_a}$ and $c_b = \sqrt{\tau/\mu_b}$, and the impedances,

$$\begin{aligned} Z_a \equiv \mu_a c_a &= \sqrt{\tau\mu_a} \\ Z_b \equiv \mu_b c_b &= \sqrt{\tau\mu_b}. \end{aligned} \tag{16}$$

We also require the wave vector in the heavy region, which we determine from the fact that the response of the system in all regions will be at the same frequency as the incoming wave. Thus, $c_a k_a = \omega = c_b k_b$, so that

$$k_b = \frac{c_a}{c_b} k_a = \left(\sqrt{\frac{\mu_b}{\mu_a}}\right) k_a. \tag{17}$$

Finally, we will need the following scattering amplitudes,

$$\begin{aligned} R_{b\to a} &= \frac{Z_b - Z_a}{Z_a + Z_b} \\ T_{a\to b} &= \frac{2Z_a}{Z_a + Z_b} \\ T_{b\to a} &= \frac{2Z_b}{Z_a + Z_b} \end{aligned} \tag{18}$$

With the above quantities defined, we can now compute the transmission amplitude for passing through the barrier. Following (12), we must consider all possible histories contributing to transmission through the barrier. The first three of these appear in the figure. In the first history, h_1, the wave transmits from string a to b, picking up a transmission amplitude factor $T_{a\to b}$, propagates across from $x = 0$ to $x = b$, picking up a phase factor $p \equiv e^{ik_b b}$, and finally transmits from string b to string a, picking up a final transmission amplitude factor $T_{b\to a}$. The first contribution to the transmitted wave is thus $T_{a\to b}pT_{b\to a}$.

The next contribution, h_2, comes from when the wave transmits from a to b, propagates across, but then reflects at the interface from b to a at $x = b$, picking up a new factor of $R_{b\to a}$. This wave then propagates back across from $x = b$ to $x = 0$, picking up the same phase factor $p \equiv e^{ik_b b}$ as before because the distance propagated is the same. At $x = 0$, the wave then reflects again with a factor $R_{b\to a}$, propagates back across the barrier with a factor of p, and finally transmits from b into a with a factor of $T_{b\to a}$. The contribution from this history is the product of all of these factors, $T_{a\to b}pR_{b\to a}pR_{b\to a}pT_{b\to a}$.

There is then a third contribution, h_3, which involves yet another ricochet in between the barriers. After this contribution, there is actually an *infinite* sequence of terms $h_{n>3}$, each involving one more ricochet than the previous term in the sequence.

Combining all of these terms, we thus write (12) for this case as

$$\begin{aligned}
Q(b) &= Q(0)\sum_h \left(\prod_{e\in h} a_e\right) \\
&= Q(0)\left[h_1 + h_2 + h_3 + \ldots\right] \\
&= Q(0)\left[T_{a\to b}pT_{b\to a} + T_{a\to b}\left(pR_{b\to a}pR_{b\to a}\right)pT_{b\to a} + T_{a\to b}\left(pR_{b\to a}pR_{b\to a}\right)\left(pR_{b\to a}pR_{b\to a}\right)pT_{b\to a} + \ldots\right] \\
&= Q(0)T_{a\to b}pT_{b\to a}\left[1 + \left(pR_{b\to a}pR_{b\to a}\right) + \left(pR_{b\to a}pR_{b\to a}\right)^2 + \ldots\right] \\
&= Q(0)\frac{T_{a\to b}pT_{b\to a}}{1 - \left(pR_{b\to a}pR_{b\to a}\right)},
\end{aligned}$$

where in the last step we use the calculus result for the sum of an infinite geometric series $1 + \underline{r} + \underline{r}^2 + \ldots = 1/(1 - \underline{r})$. Note that in the analysis above, we could at any step have simplified $pR_{b\to a}pR_{b\to a} = p^2R_{b\to a}^2$. We have chosen not to do so only as a way of reminding ourselves of the sequence of events underlying each term in the series.

Finally, the net transmission amplitude for the barrier, $T_{\text{barrier}} \equiv Q(b)/Q(0)$, is thus

$$T_{\text{barrier}} = \frac{T_{a\to b}pT_{b\to a}}{1 - p^2R_{b\to a}^2}, \tag{19}$$

where $p \equiv e^{ik_b b}$ and all other relevant quantities are defined in (16-18).