Class Notes IV: General Solution to Wave Equation — Traveling Waves, Superposition and Reflection

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1 Introduction

The previous set of notes illustrated the ubiquity of wave behavior by showing that the displacements in strings and sound (y and s, respectively) and that the electromagnetic fields in vacuum (E_y, B_y, E_z, B_z) all satisfy the same wave equation,

$$\frac{\partial^2 q}{\partial t^2} = c^2 \frac{\partial^2 q}{\partial x^2}, \tag{1}$$

where q represents either y, s or E_y (or B_y, E_z, or B_z) for strings, sound or electromagnetic systems, respectively. We now turn to the problem of finding a general solution to this equation. Once we have such a solution, we are then able to explore readily all possible wave solutions and therefore all possible wave phenomena.

Along the way to the general solution, we shall also discover a new, important sub-class of solutions to the wave equation, *traveling waves*[1]. These traveling solutions will clarify greatly why we interpret the constant c in the wave equation (1) to be the *wave speed*.

2 Pulse equations

To find a general solution to an equation such as (1), it generally behooves us to simplify it first. The wave equation (1) appears as though it could be simplified by taking its "square root." In solving any kind of equation, we certainly always are allowed to take a guess at a simplification, so long as we check that it is correct in the end. Our guess at a "square root" is the following equation, which we shall call the *pulse* equation,

$$\frac{\partial q}{\partial t} = \mp c \frac{\partial q}{\partial x} \quad \text{Pulse equation.} \tag{2}$$

Note that in taking the "square root", we have been careful to include both possible choices of sign so that there are really two pulse equations, one for each sign.

To check that we have made a proper simplification, we now verify that any solution to either pulse equation (either the "−" or the "+" version) is also guaranteed to solve the wave equation. To do this, we are allowed to assume at each step that we satisfy the pulse equation (2) and try to prove that q satisfies the wave equation (1):

$$\begin{aligned}
\frac{\partial^2 q}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\mp c \frac{\partial q}{\partial x}\right) \quad \text{; Pulse Eq} \\
&= (\mp c)\frac{\partial}{\partial x}\left(\frac{\partial q}{\partial t}\right) \quad \text{; Switch order of derivs} \\
&= (\mp c)\frac{\partial}{\partial x}\left(\mp c \frac{\partial q}{\partial x}\right) \quad \text{; Pulse Eq (again!)} \\
&= (\mp c)(\mp c)\frac{\partial^2 q}{\partial x^2} \\
&= c^2 \frac{\partial^2 q}{\partial x^2}, \quad \text{; Solves Wave Eq!}
\end{aligned}$$

where we have used the fact that $(-1)(-1) = (1)(1) = 1$. The fact that we were able to prove that $q(x,t)$ solves the wave equation by assuming only that it solves either pulse equation means that, indeed, if we ever have a solution to the pulse equation we know also automatically that it solves the wave equation!

3 General solutions to pulse equations

We now consider what sorts of functions satisfy the pulse equation. Apart from the factor of $\mp c$, the pulse equation (2) says that the t-derivative of $q(x,t)$ gives exactly the same result as the x-derivative. A good guess for a solution is some function of $x + t$ because then, when we apply the chain rule, we will get the same function back times either $\partial/\partial x$ or $\partial/\partial t$ of $(x+t)$, which are both just one. Actually, however, the pulse equation says that we want the time derivative to give back an extra factor of $\mp c$. Thus, a guess for a very general type of solution to the pulse equation would be

$$q(x,t) = f(x \mp ct), \quad \text{; general solution to (2)} \tag{3}$$

where $f()$ is *any function whatsoever!*

[1] We already explored one other type of solution in Class Notes II, standing waves.

To verify the above as a solution, we take the appropriate derivatives and check the pulse equation (2),

$$\begin{aligned}\frac{\partial q}{\partial x} &= \frac{\partial}{\partial x} f(x \mp ct) \\ &= f'(x \mp ct) \frac{\partial}{\partial x}(x \mp ct) \\ &= f'(x \mp ct) \cdot 1 \\ &= f'(x \mp ct). \end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\partial q}{\partial t} &= \frac{\partial}{\partial t} f(x \mp ct) \\ &= f'(x \mp ct) \frac{\partial}{\partial t}(x \mp ct) \\ &= f'(x \mp ct)\,(\mp c) \\ &= (\mp c) f'(x \mp ct) \\ &= (\mp c)\frac{\partial q}{\partial x}, \quad \text{; using (4)}\end{aligned}$$

which proves that our form (3) for $q(x,t)$ indeed solves (2). Moreover, because of our discussion in Section 2, we know that these solutions also solve the wave equation!

To interpret these solutions, consider Figure 1. Figure 1a shows the solution at time $t = 0$, which is just a picture of the function $f()$ because $q(x, t = 0) = f(x \mp c \cdot 0) = f(x)$. Figure 1b shows the solution for the "$-$" version of the pulse equation at some time t later: $q(x, t = 0) = f(x - c \cdot t) = f(x - a)$, where we have defined $a \equiv ct$. As the picture illustrates, this is just exactly the same shape but shifted to the *right*[2] by the distance $a = ct$. Similarly, Figure 1c shows the solution for the "$+$" version of the equation at some time t later: $q(x, t = 0) = f(x + c \cdot t) = f(x + a)$, where we again have defined $a \equiv ct$, which is just the same shape but shifted rigidly to the *left*[3] by $a = ct$. Thus, our solutions to the the "$-$" and "$+$" versions of the pulse equation represent *pulses*, shapes which shift rigidly to the right or left, respectively, with constant speed $v = a/t = (ct)/t = c$. This is precisely why we have always referred to c as the *wave speed* and why we refer to (2) as the left/right pulse equations.

Given that our solutions (3) to the left/right pulse equations are in terms of a completely undetermined function $f()$, they are indeed quite general. But, are they truly *general solutions*? As discussed in Class Notes I, " 'Simple Harmonic' Motion (SHM)," a general solution not only solves the equation of motion, but also does so with a number of adjustable parameters (unspecified constants which may take any value) equal to one for each order of time derivative appearing for each degree of freedom. The equation of motion for the simple harmonic oscillator was second-order in time, and therefore the general solution requires two adjustable parameters for each degree of freedom. As there is only one degree of freedom, "x", the general solution for the simple harmonic oscillator has two adjustable parameters.

The pulse equations, on the other hand, are first-order in time. Therefore, a general solution must have one adjustable parameter for each degree of freedom. In the pulse equations, as in the wave equation, the degrees of freedom are the values $q(x)$ which specify the configuration of the system (position of a string, displacement of the air, value of the electric field, *etc.*) at any given instant in time. Now, $q(x)$, which could be any function of x, actually specifies a large number of values, one value of q for each value of x. There is therefore one degree of freedom for each point x. Correspondingly, for a general solution, we need one adjustable parameter for each point x. Note that we emphasized that $f()$ in (3) could be any function *whatsoever*. This means that we can pick any value of $f(x)$ for each point x and still have a solution. The values $f(x)$ for each x are, therefore, adjustable, and we have precisely one adjustable parameter, the value $f(x)$ for each x, for each degree of freedom. The general solutions to (2) are therefore (3).

[2] To see that $f(x - a)$ is shifted to the right, note that the value $f(0)$, which occurred at $x = 0$, now occurs at $x = a$.
[3] To see that $f(x + a)$ is shifted to the left, note that the value $f(0)$, which occurred at $x = 0$, now occurs at $x = -a$.

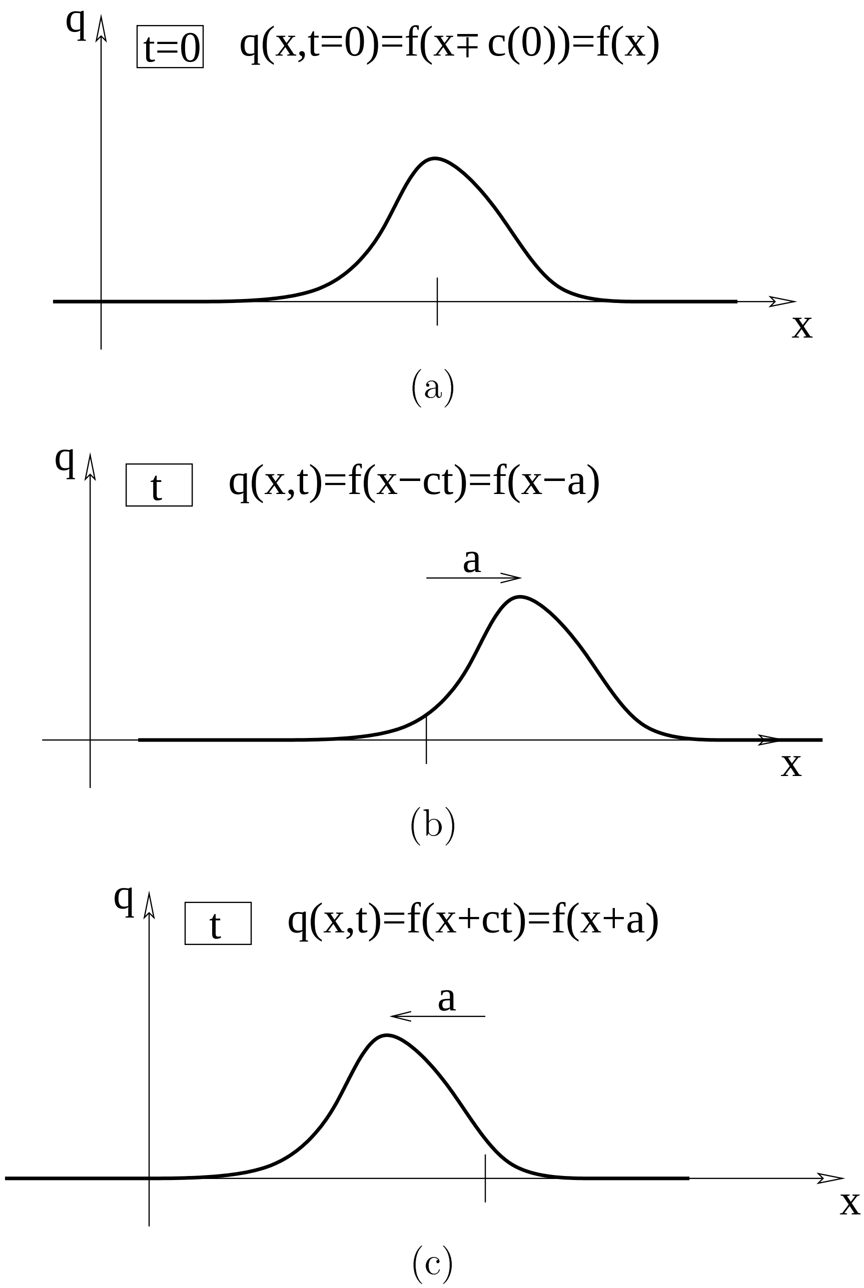

Figure 1: Time evolution of solutions to the pulse equations: (a) solution at $t = 0$, (b) right-ward or "$-$" solution at time $t > 0$, (c) left-ward or "$+$" solution at time $t > 0$.

4 Superposition and general solution to wave equation

To find the general solution to the wave equation, we first prove an important lemma which has many useful applications in the theory of waves, the principle of superposition. Much like the principle of superposition which you learned in your introductory course on E&M, this principle allows us to quickly solve complex problems by breaking them into smaller parts for which we know the solutions.

> *Principle of superposition of waves* — If $y_1(x,t)$, $y_2(x,t)$, ... all solve the wave equation, then the sum $y(x,t) = y_1(x,t) + y_2(x,t) + \ldots$ is another valid solution.

Proof: To verify that $y(x,t)$ is a solution, we compute $\partial^2 y/\partial t^2$ and check whether it indeed equals $c^2\partial^2 y/\partial x^2$. During our computation, we are allowed to use the facts that $\partial^2 y_1/\partial t^2 = c^2\partial y_1/\partial x^2$, $\partial^2 y_2/\partial t^2 = c^2\partial y_2/\partial x^2$, ..., because the assumption of the lemma is that y_1, y_2, ..., are all solutions. The result is

$$\begin{aligned}
\frac{\partial^2 y(x,t)}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left(y_1(x,t) + y_2(x,t) + \ldots\right) && \text{; definition of } y(x,t) \\
&= \frac{\partial^2}{\partial t^2} y_1(x,t) + \frac{\partial^2}{\partial t^2} y_2(x,t) + \ldots && \text{; sum rule for derivatives} \\
&= \left(c^2 \frac{\partial^2}{\partial x^2} y_1(x,t)\right) + \left(c^2 \frac{\partial^2}{\partial t^2} y_2(x,t)\right) + \ldots && \text{; } y_1(x,t),\ y_2(x,t),\ \ldots, \text{ are solutions} \\
&= c^2\left(\frac{\partial^2}{\partial x^2} y_1(x,t) + \frac{\partial^2}{\partial t^2} y_2(x,t) + \ldots\right) && \text{; factoring out } c^2 \\
&= c^2 \frac{\partial^2}{\partial x^2}\left(y_1(x,t) + y_2(x,t) + \ldots\right) && \text{; sum rule for derivatives} \\
&= c^2 \frac{\partial^2}{\partial x^2} y(x,t), && \text{; definition of } y(x,t)
\end{aligned}$$

and so $y(x,t)$ indeed solves the wave equation.

To find a *general* solution to the wave equation, we note that the solutions to the pulse equation $f(x \mp ct)$ both automatically solve the wave equation[4], but each only has one adjustable parameter for each degree of freedom. The wave equation is second-order in time and therefore requires *two* adjustable parameters for each degree of freedom. Thus, although each general solution to the pulse equation has insufficient adjustable parameters to be a general solution to the wave equation, by superposing two pulses, one left-ward and one right-ward,

$$y(x,t) = f(x - ct) + g(x + ct), \tag{5}$$

we are guaranteed to have a general solution to the wave equation. First, by superposition, we know we have a solution. Second, we will then have two adjustable parameters for each degree of freedom, one value of $f(x)$ and one value of $g(x)$ for each value of x.

5 Finding particular solutions

As with the simple harmonic oscillator, the great power of having the general solution is that we now can find the particular solution for any problem without solving complicated differential equations but by solving for the adjustable parameters in the general solution. The basic strategy is the same as with the harmonic oscillator: "(1) write each condition in terms of the general solution, (2) solve the resulting set of equations for the adjustable parameters, and (3) write down the general solution while substituting the particular values found for the adjustable parameters."[5]

Although the strategy is the same, following this procedure is more challenging in the case of waves, particularly because the "adjustable parameters" are actually now unknown *functions*, $f()$ and $g()$. We therefore shall work through two different examples: first, when we are given the initial position and initial velocity of every chunk of the system, and, second, when a pulse collides with a boundary.

[4] Recall that any solution to the pulse equation also solves the wave equation.
[5] From "Class Notes I".

5.1 Particular solutions from initial conditions

Suppose that we are given the value and velocity of the wave at time $t = 0$: $q(x, t = 0) = q_0(x)$ and $\partial q(x, t = 0)/\partial t = v_0(x)$[6], and we wish to determine $q(x, t)$ the form of the wave at any time t later. To do this we follow the three-step procedure above.

1. *Write each condition in terms of the general solution —*

   The general solution is
$$q(x, t) = f(x - ct) + g(x + ct). \tag{6}$$
   In terms of this, the first initial condition is
$$\begin{aligned} q_0(x) = q(x, t = 0) &= f(x - c \cdot 0) + g(x + c \cdot 0) \\ &= f(x) + g(x), \end{aligned} \tag{7}$$
   and so we learn that the sum of the two unknown functions gives us the initial configuration of the system. The second condition is
$$\begin{aligned} v_0(x) = \frac{\partial}{\partial t} q(x, t = 0) &= \left((-c)f'(x - ct) + (c)g'(x + ct)\right|_{t=0} \\ &= (-c)f'(x - c \cdot 0) + (c)g'(x + c \cdot 0) \\ &= c\left(g'(x) - f'(x)\right), \end{aligned} \tag{8}$$
   and so we learn that the wave speed times the difference of the derivatives of the two unknown functions gives us the initial velocities throughout the system.

   Interpretation of the two results (7,8) sometimes causes confusion because the variable t appears in the general solution (6) but not in the results. The best way to understand our results is to think of (7,8) as giving relationships between the various adjustable parameters, values of the the functions $f()$ and $g()$. Eq. (7), for instance, says that when adding the values $f(3)$ and $g(3)$, one gets $q_0(3)$, and so forth for all possible values. Speaking very generally, and not thinking specifically about points in space, we could have used any value u in place of the number 3. Thus, a less confusing way of thinking about (7) would be to write
$$q_0(u) = f(u) + g(u), \tag{9}$$
   so that we don't get hung up thinking about positions in space x or time t. Similarly, to avoid confusion, it is best to write (8) as
$$v_0(u) = c\left(g'(u) - f'(u)\right). \tag{10}$$

2. *Solve the resulting set of equations for the adjustable parameters —*

   For waves, the functions $f()$ and $g()$ give the adjustable parameters, and so for this step, we must solve (9) and (10) for $f(u)$ and $g(u)$.

   We begin by simplifying (10). Integrating both sides with respect to u, we find
$$\begin{aligned} \int v_0(u)\,du + C &= c\left(g(u) - f(u)\right) \\ \frac{1}{c}\int v_0(u)\,du + \frac{C}{c} &= g(u) - f(u), \end{aligned} \tag{11}$$
   where C is a constant of integration and $\int v_0(u)\,du$ is the anti-derivative of the function $v_0(u)$.

[6] For concreteness, it may help to think of the example of the string. For the string (as with any mechanical system), you generally need to know both the location $y_0(x)$ and velocity $v_0(x)$ of each particle (chunk) at time $t = 0$ in order to predict where all of the particles will be at any time t in the future.

Now, by adding and subtracting, respectively, the two equations (9) and (11), we find

$$\begin{aligned} q_0(u) + \frac{1}{c}\int v_0(u)\,du + \frac{C}{c} &= 2g(u) \\ q_0(u) - \frac{1}{c}\int v_0(u)\,du - \frac{C}{c} &= 2f(u). \end{aligned}$$

Finally, dividing through by 2, we have the two unknown functions,

$$\begin{aligned} f(u) &= \frac{1}{2}\left(q_0(u) - \frac{1}{c}\int v_0(u)\,du\right) - \frac{C}{2c} \\ g(u) &= \frac{1}{2}\left(q_0(u) + \frac{1}{c}\int v_0(u)\,du\right) + \frac{C}{2c}. \end{aligned} \tag{12}$$

3. *Write down the general solution while substituting the particular values found for the adjustable parameters —*

   Substituting the results (12) into the general solution, we find

$$\begin{aligned} q(x,t) &= f(x-ct) + g(x+ct) \\ &= f(u)|_{u=x-ct} + g(u)|_{u=x+ct} \\ &\Rightarrow \\ q(x,t) &= \frac{1}{2}\left(q_0(u) - \frac{1}{c}\int v_0(u)\,du\right)\bigg|_{u=x-ct} + \frac{1}{2}\left(q_0(u) + \frac{1}{c}\int v_0(u)\,du\right)\bigg|_{u=x+ct}, \end{aligned} \tag{13}$$

   where the unknown integration constant C has canceled out conveniently.

The result (13) is somewhat abstract. To learn how to use it, consider an example of a string which is initially "flat" ($y(x,t=0) = 0$ for all x), but which has been given an initial "kick" near the origin so that the initial velocity distribution is $v_y(x,t=0) = w/(1+(x/a)^2)$. Now, to use (13), we need the following integral,

$$\begin{aligned} \int v_0(u)\,du &= \int v_y(u,t=0)\,du \\ &= \int \frac{w}{1+(u/a)^2}\,du \\ &= a\int \frac{w}{1+(u/a)^2}\,\frac{du}{a} \quad \text{; preparing for change of variables} \\ &= a\int \frac{w}{1+z^2}\,dz \quad ;\ z \equiv u/a \\ &= aw\arctan z \\ &= aw\arctan\left(\frac{u}{a}\right) \quad \text{; using def. of } z. \end{aligned} \tag{14}$$

We also need the initial displacement, which was just zero in this case, $q_0(u) = 0$. Substituting this and the integral (14) into the solution (13), we get the final result,

$$\begin{aligned} q(x,t) &= \frac{1}{2}\left(q_0(u) - \frac{1}{c}\int v_0(u)\,du\right)\bigg|_{u=x-ct} + \frac{1}{2}\left(q_0(u) + \frac{1}{c}\int v_0(u)\,du\right)\bigg|_{u=x+ct} \\ &= \frac{1}{2}\left(0 - \frac{1}{c}\,aw\arctan\left(\frac{u}{a}\right)\right)\bigg|_{u=x-ct} + \frac{1}{2}\left(0 + \frac{1}{c}\,aw\arctan\left(\frac{u}{a}\right)\right)\bigg|_{u=x+ct} \\ &= \frac{aw}{2c}\left(\arctan\left(\frac{x+ct}{a}\right) - \arctan\left(\frac{x-ct}{a}\right)\right). \end{aligned}$$

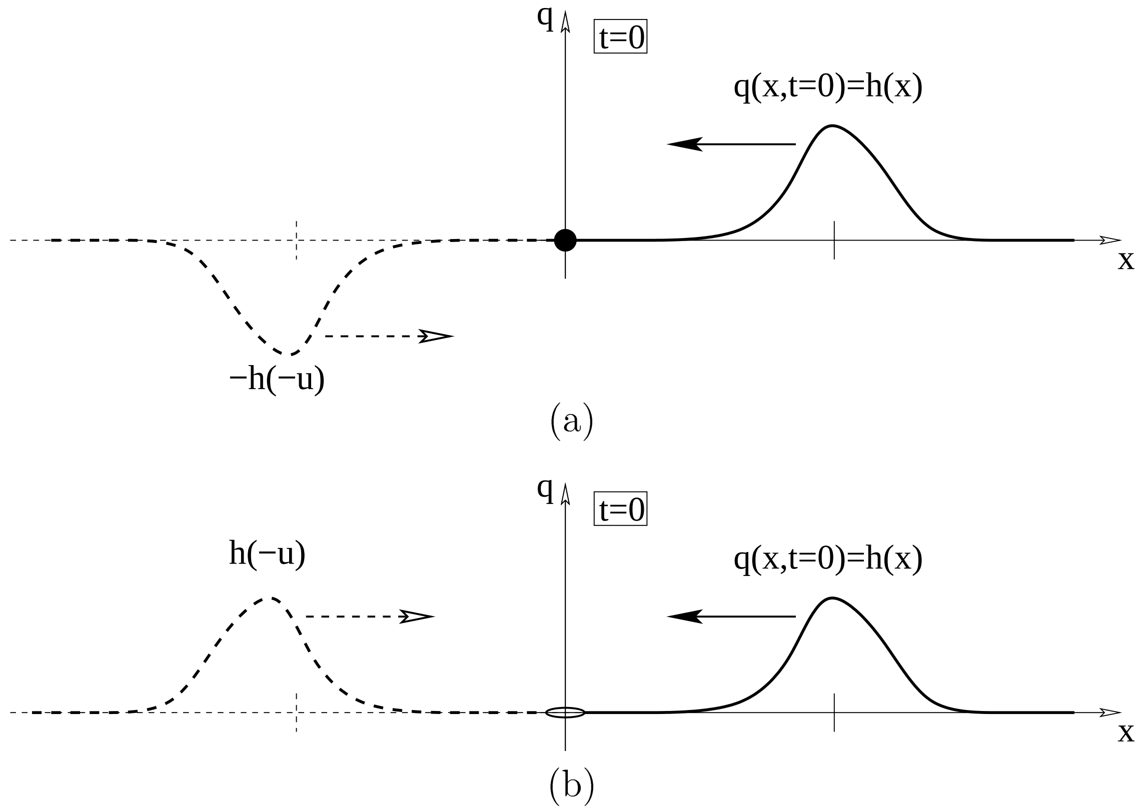

Figure 2: Reflection of an incoming left-ward pulse of shape $h(x)$ from a boundary at $x = 0$: (a) fixed boundary, (b) free boundary

5.2 Particular solution for reflection from boundaries

As another example of use of the general solution, we now consider an important wave phenomenon, reflection from a boundary. Consider the situations in Figures 2a and b, in which an initial left-ward pulse of shape $h(x)$ heads toward a boundary at $x = 0$ with either fixed ($q(x = 0, t) = 0$) or free ($\partial q(x = 0, t)/\partial t = 0$) boundary conditions, respectively.

In order to determine what the system will do at later times, we use the fact that the general solution, by definition, describes *all* possible behaviors of the string. Thus, we know that the equation

$$q(x, t) = f(x - ct) + g(x + ct) \tag{15}$$

describes behavior of the string at any later time t. (Note that whatever this equation gives for $x < 0$ is just a mathematical fiction as the system ends at the boundary at $x = 0$.) Our task, therefore, reduces to finding the values of the unknown functions $f(u)$ and $g(u)$ at all possible values of their argument u. Again, we use "u" instead of "x" to remind ourselves that the values $f(u)$ and $g(u)$ for each u really are unknown parameters and not tied to any particular point in space or time.

To find the unknown functions, we again use the initial conditions. There is now an additional condition which we must take into account, however. Recall that the presence of a boundary implies that the point on the boundary does not obey the same equation of motion as the other points, but rather a special equation which we called the boundary condition. This means that the general solution to the wave equation (15) applies everywhere but on the boundary and so we must take the boundary condition

$$q(x = 0, t) \quad = \quad 0 \quad ; \text{fixed boundary at } x = 0\text{: case (a)}$$

$$\frac{\partial q(x=0,t)}{\partial x} = 0 \quad \text{; free boundary at } x=0\text{: case (b)}$$

into account explicitly as an addition condition.

The initial condition was simply the presence of a left-ward moving pulse of shape $h(x)$. This we will get if we take $g(u) = h(u)$ because the second term of our solution will then be $g(x+ct) = h(x+ct)$, a left-ward moving pulse of shape $h(u)$. We thus have left only to find $f(u)$, which we do by writing the boundary condition explicitly in terms of our general solution.

For the case of a fixed boundary, using the fact that we already know that $g(u) = h(u)$, we find

$$\begin{aligned}
0 &= q(x=0,t) \\
&= f(0-ct) + h(0+ct) \\
&= f(-ct) + h(ct) \qquad (16) \\
&= f(u) + h(-u) \quad \text{; change of variables } u \equiv -ct \\
&\Rightarrow \\
f(u) &= -h(-u). \qquad (17)
\end{aligned}$$

To understand the meaning of the change of variables, note that Eq. (16) just says, for instance, that $0 = f(-3) + h(3)$, and so, therefore, $f(-3) = -h(3)$. Or that $0 = f(5) + h(-5)$, and thus $f(5) = -h(-5)$. The change of variables just allows us to say that this holds for all values, not just -3 and 5.

The interpretation of (17) is that, for this case of a fixed boundary, there is also a right-ward moving pulse (*i.e.*, a *reflection*) which the boundary generates. The shape of the reflection, as Figure 2a illustrates, is just the same as that of the incoming pulse, *except* that it is up-side down (the first minus sign in (17)) and left-right inverted (the second minus sign in the equation). To complete the solution for the case of a fixed boundary, we substitute our results for $f(u)$ and $g(u)$ into the solution (15). In doing this, we note that (17) says that when we want the value of $f(u)$ at any point u, whether u be -3, 5 or $x - ct$, we just put $-h(-u)$. Thus, the solution for reflection from a fixed boundary is

$$q(x,t) = h(x+ct) - h\left(-(x-ct)\right). \quad \text{; reflection from a fixed boundary} \qquad (18)$$

The case of a free boundary is similar, but somewhat more complicated because of the derivative in the boundary condition. Following the same procedure of substituting our solution into the boundary condition, we find

$$\begin{aligned}
0 &= \frac{\partial q(x=0,t)}{\partial x} \\
&= \frac{\partial}{\partial x}\left(f(x-ct) + h(x+ct)\right)\Big|_{x=0} \\
&= \left(f'(x-ct)\frac{\partial(x-ct)}{\partial x} + h'(x+ct)\frac{\partial(x+ct)}{\partial x}\right)\Bigg|_{x=0} \quad \text{; chain rule} \\
&= \left(f'(x-ct)\cdot 1 + h'(x+ct)\cdot 1\right)\big|_{x=0} \quad \text{; taking partial derivs} \\
&= \left(f'(0-ct) + h'(0+ct)\right) \quad \text{; evaluating at } x=0 \\
&= \left(f'(-ct) + h'(ct)\right) \qquad (19) \\
&= \left(f'(u) + h'(-u)\right) \quad \text{; change of variables } u \equiv -ct \\
&\Rightarrow \\
f'(u) &= -h'(-u). \qquad (20)
\end{aligned}$$

Here, we again make the convenient substitution that $u = -ct$. Finally, we must solve (20) for the unknown function $f(u)$. This we do by taking the anti-derivative,

$$f(u) = h(-u) + C,$$

which is easily checked by taking the derivative with respect to u and recovering (20). To determine the unknown integration constant C, we can play our usual trick and note that $h(\pm\infty) = 0$ and that we also expect $f(\pm\infty) = 0$, and so, when we evaluate our solution for $f(u)$ at either of these points, we find that $C = 0$. Thus, for free boundary conditions, we conclude that the reflected pulse is

$$f(u) = h(-u), \tag{21}$$

which means that the reflection, as Figure 2b illustrates, is now right-side up (no initial minus sign) and still left-right reflected (minus sign inside $h()$). Once again, we substitute into (15) to get the final solution for reflection from a free boundary,

$$q(x,t) = h(x+ct) + h\left(-(x-ct)\right). \quad \text{; reflection from free boundary} \tag{22}$$