Class Notes II: Introduction to waves

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Contents

List of Figures

1 Introduction

We now begin our study of waves. Note that we follow precisely the same outline of attack which we had for Simple Harmonic Motion. We begin with a simple physical realization, waves on a string (Section 2). We then proceed to use this example to introduce our basic characterization of wave behaviors in Section 3, and then begin the general analysis in Section 4.

After using waves on a string to introduce a number of basic wave concepts, to identify the relevant degrees of freedom, and to derive the equation of motion, we shall conclude this set of notes by investigating

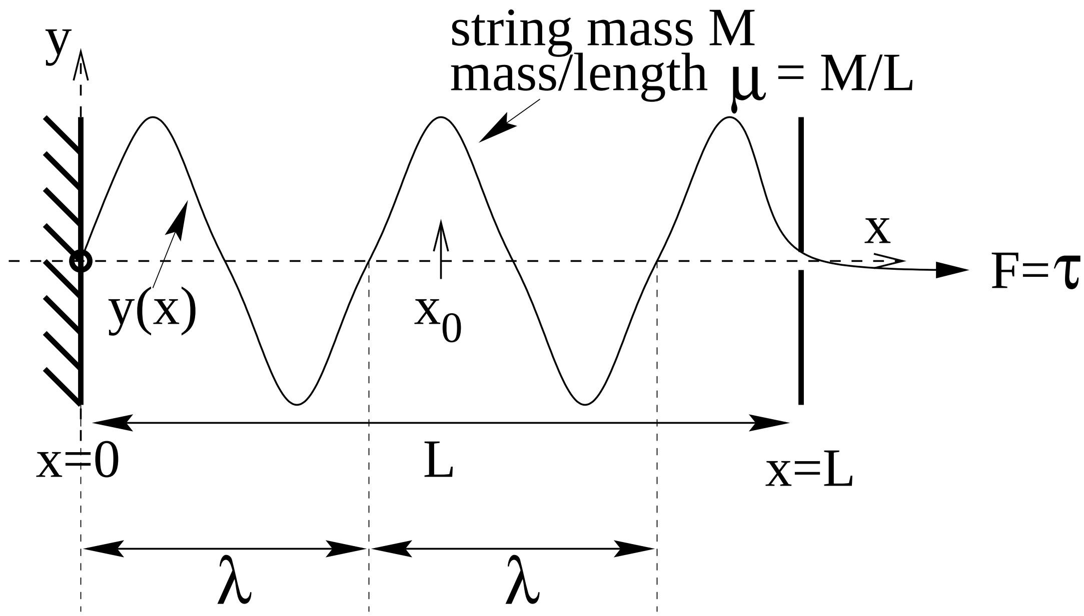

Figure 1: String system illustrating simple wave behavior

some solutions to the wave equation. Our intention is then to investigate other systems exhibiting wave behavior before returning to finding the general solution to the wave equation.

2 Physical realization

For our first physical realization of a system exhibiting wave behavior, we use the string system in Figure 1. The string stretches between two vertical walls held at a distance L apart, one wall at $x = 0$ and the other at $x = L$. The wall at $x = 0$ holds the string in place at $y = 0$, where the string attaches to a pin. The wall at $x = L$ has a small hole at $y = 0$ through which the string passes. This ensures that the string at this end also is at position $y = 0$ while allowing us to apply a force F to keep the string under constant tension τ. Otherwise, the string is free to move. The figure shows an example where the string has been “plucked” so that it now supports a wave as sketched in the figure. The total mass of the length of string between the walls is M and, thus, the mass per unit length of the string is $\mu \equiv M/L$.

3 Basic Characterization

The second step in understanding a new phenomena is to identify the basic quantities which we hope to describe and understand. Under appropriate conditions (demonstrated in lecture), vibrations exist where each little segment of the string in Figure 1 moves up and down periodically. Wave motion often involves such periodic motion in time, something whose characterization we reviewed in the previous set of notes. Note that also that the shape of the string shown in the Figure 1 repeats every time we move a distance λ down the string. Thus, waves exhibit periodic behavior not only in time but also *in space*.

3.1 Descriptions of Spatial Periodicity

We now develop a new vocabulary to describe this new kind of periodicity. It will be directly analogous to our description of periodicity in time. The only difference is in terminology and the fact that we use measures of distances instead of times.

- *Wavelength* λ: The distance we must travel down the wave for a full period or cycle. Wavelength is typically measured in meters, so that the basic unit is 1 **m**.
- *Wavenumber* κ: The number of cycles or waves which occur in a unit distance. The unit of wavenumber is typically 1 **cycle/m**.
- *Wave-vector* k: The number of radians of phase which we cross in a unit distance, where we associate 2π radians of phase with each cycle. The unit of angular frequency is thus typically **1 radian/m**. Because radians carry no dimension, we frequently omit the radian and write the unit of wave vector as simply 1 $\mathbf{sec}^{-1}$.

3.2 Conversions

As with the measures of temporal periodicity, all three spatial measures can be converted among each other. It is important to be able to do this quickly.

If one cycle takes a distance λ, then the number of cycles which occur per unit distance is

$$\kappa = \frac{1\ \mathbf{cycle}}{\lambda}. \tag{1}$$

Because there are 2π radians in one cycle, the number of radians which we cross per unit distance are therefore

$$k = \frac{2\pi\ \mathbf{rad}}{\lambda} = \frac{2\pi}{\lambda}, \tag{2}$$

where we remind ourselves that we may drop the optional "unit" **rad**. Finally, combining (1-2), we have

$$k = 2\pi\kappa. \tag{3}$$

3.3 Dispersion Relation

Given the conversions in Section 3.2 and the conversions among temporal quantities from the previous set of notes, we can convert any temporal quantity into any other and any spatial quantity into any other. What we lack is an ability to link one of the spatial quantities into one of the temporal quantities. Then, we can convert any of the six quantities T, f, ω, λ, κ, k to any other. A relation linking a spatial and temporal quantity is known as a *dispersion relation*.

To generate one such relation, consider the wave illustrated in Figure 1. Suppose that the wave travels to the right with *wave speed* v. The wave speed does not necessarily refer to the speed of individual parts of the string, but rather to the speed at which the pattern of the wave moves.

As the wave passes, the point at location x_0 first moves down into the trough of the oncoming wave and then comes back up as the crest of the next wave comes directly under the point. This up-down-up process represents one full period and takes time T. During this time, the crest of the oncoming wave has moved precisely one repeat distance, or wavelength λ. The speed of the wave, *wave speed* v, is therefore

$$v = \frac{\lambda}{T}. \tag{4}$$

As this relates one of the spatial and one of the temporal quantities together, Eq. 4 qualifies as a dispersion relation and allows us to convert among any of the six basic wave quantities provided that we know the value of the wave speed v. To determine this speed for the string, we now proceed to the detailed analysis of the motion of the string.

4 General Analysis

4.1 Identify the degrees of freedom

The first step in the general analysis is to identify the degrees of freedom, the minimal set of variables needed to define the configuration of the system at a given instant in time. Because the string consists of a large collection of particles or segments, we must be able to specify the location of each such segment. For simplicity, we here consider only up and down motions of the string, thus the degrees of freedom must specify the x and y coordinates of each segment.

Simplifying approximation — We shall make one single simplifying approximation in our analysis of the string. We shall assume that the amplitude of the waves on the string is small. One could begin without this approximation, carry the analysis in full and then take the limit of small amplitudes. It turns out, however, that in practice the small amplitude approximation is quite accurate even for relatively large vibrations and that the full analysis becomes quite complicated. Without this approximation, we would learn little for much additional effort. Thus, we shall assume from this point onward we shall assume that the amplitude of the waves we study is small.

This approximation has a very important implication of which we shall make much use. If the amplitude of the waves on the string is small, then the length of string between the walls is always very nearly L. Thus, there is very little pulling of the string in and out of the hole where we apply the tension and the x position of each segment changes negligibly. As the x positions of the segments do not change, they do not need to be specified as degrees of freedom. Moreover, we can use the x position as a way to identify or label the segments.

To specify the state of the string we need only specify the y-location of each segment at horizontal location x between the walls. Mathematically, this is the same a giving a function $y(x)$. This specifies the state of the string because to determine what the string looks like, one would simply plot the given function $y(x)$.

Given $y(x)$ as a way to specify the degrees of freedom, a solution for the string should specify a function $y(x)$ for each value of time t. Mathematically, this is the same as a function of the form $y(x,t)$ because, for any given instant in time t_0, we can get the state of the string by plotting $y(x,t_0)$ versus x. Note that because a solution $y(x,t)$ is a function of two variables, we shall now be working primarily with *partial* derivatives.

4.2 Derive the equation of motion

To derive the equation of motion, we express Newton's law of motion for each particle (string segment) solely in terms of the solution q$y(x,t)$ and constants specified in the problem. We therefore consider the free-body diagram for a single individual segment, which we draw in Figure 2 for the segment of string between positions x and $x+\Delta$.

Only two forces act on a given string segment, the tensions from the segments neighboring to the left and eight: no significant long-range forces act[1] and the only other contact with the string is with the surrounding air[2]. These tension forces act along the direction tangent to the string. Anticipating coming developments, the figure brakes the tension forces into x and y components. The tensions on either side of the segment need not be equal[3], and so we further identify the components of the tension as being measured either at point x ($T_x(x)$ and $T_y(x)$) or at point $x+\Delta$ ($T_x(x+\Delta)$ and $T_y(x+\Delta)$).

Newton's Law for the states

$$\sum \vec{F}_{\text{ext}} = m_{ch}\vec{a}_{\,\text{c of m}}, \tag{5}$$

where we use the version of Newton's law which applies to finite bodies so that we need not assume that the segment is a small point. In this version of Newton's law, we need only consider external forces acting on the system, namely the tensions acting on either end, $\vec{a}_{\,\text{c of m}}$ is the acceleration of the center of mass of the segment, and m_{ch} is the mass of the entire segment. Because the mass per unit length of the string is μ and the length of the segment is Δ[4], we have $m_{ch} = \mu\Delta$.

[1] The only two classes of such forces are electromagnetic and gravitational. We do not consider electromagnetic forces. Also, the effects of the gravitational forces will be negligible compared to the tension forces so long as the string is sufficiently light (μ is small) or the tension sufficiently large (τ large). This holds whenever the the string hangs between the walls without sagging.

[2] We also ignore air resistance

[3] Tensions are only equal for massless strings.

[4] In the figure, the segment may appear to have a length significantly longer than Δ. This is because we have exaggerated the amplitude of the involved wave for clarity.

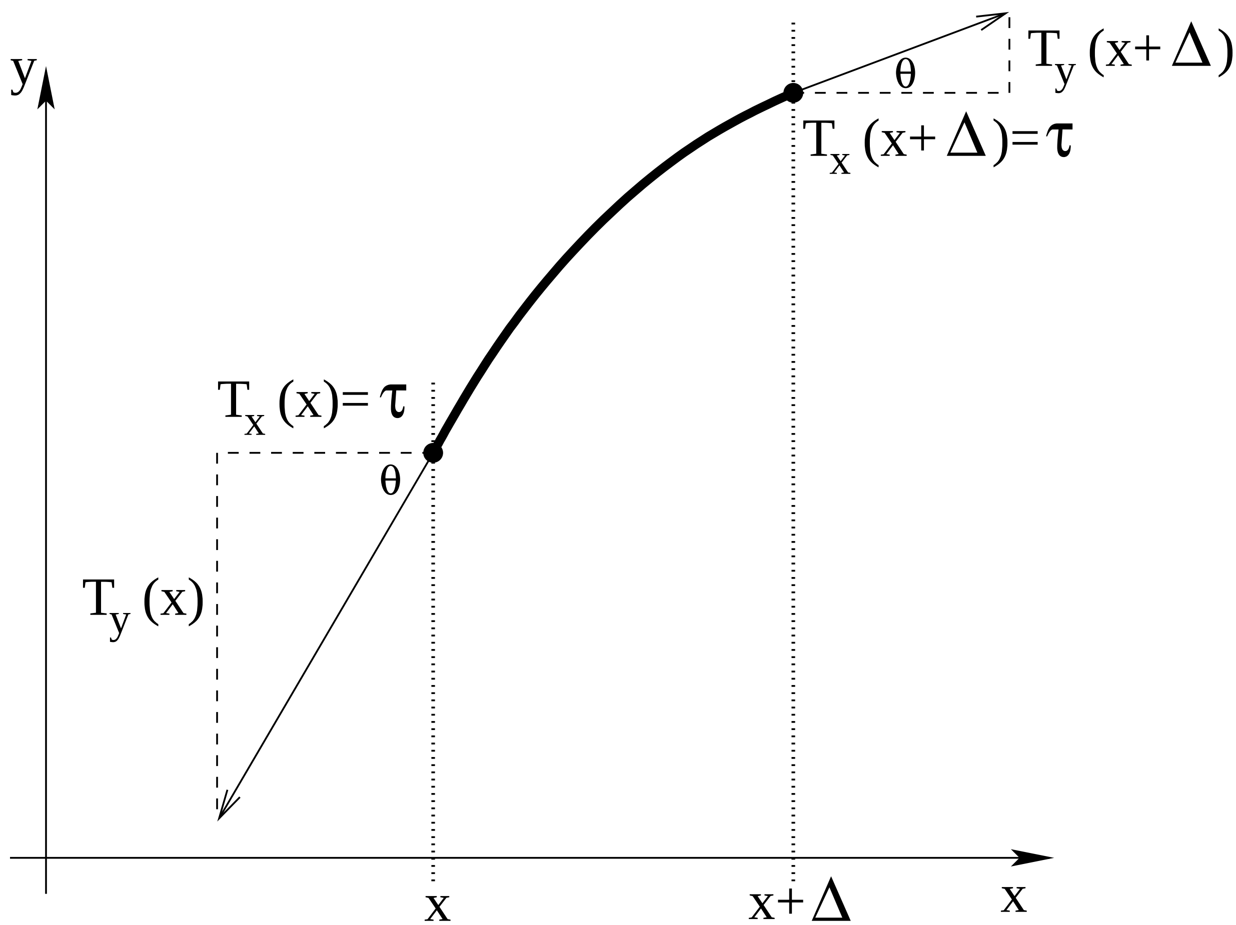

Figure 2: Free-body diagram for a segment of string

4.2.1 Motion in the x direction

The x-component of (5) gives

$$\begin{aligned}\sum F_{\text{ext},x} &= m_{ch} a_{\,\text{c of m},x} \\ +T_x(x+\Delta) - T_x(x) &= (\mu\Delta)\cdot 0 = 0 \\ &\Rightarrow \\ T_x x + \Delta &\Rightarrow T_x(x),\end{aligned} \tag{6}$$

where we have been careful to maintain the proper sign conventions on the forces, with the right-hand direction regarded as positive, and have used the fact that the segments do not move appreciably in the x-direction (low amplitude approximation) and thus $a_{\,\text{c of m},x} = 0$. Because we have been careful to use the version of Newton's law which applies to finite-sized objects, the result (6) applies to any segment. Thus, x and $x+\Delta$ could be any two points along the string, the x-component of the tension at any two points along the string are the same, and thus this component of the tension is constant throughout. The horizontal force τ which we apply to the end of the string (Figure 1) sets the value of this constant. Thus, we have as a general result

$$T_x = \text{constant} = \tau. \tag{7}$$

4.2.2 Motion in the y direction

The y-component of (5) gives

$$\begin{aligned}\sum F_{\text{ext},y} &= m_{ch} a_{\,\text{c of m},y} \\ +T_y(x+\Delta) - T_y(x) &= (\mu\Delta)\, a_{\,\text{c of m},y} \\ &\Rightarrow \\ \frac{T_y(x+\Delta) - T_y(x)}{\Delta} &= \mu a_{\,\text{c of m},y},\end{aligned}$$

where we have divided through by Δ, making the left-hand side appear like a derivative. It is now quite natural to consider the limit $\Delta \to 0$. In this limit, the left-hand side approaches the partial derivative $\partial T_y/\partial x$. (The derivative is partial because we consider the two tensions at the same instant in time and only x varies in taking the difference.) As we take the the limit $\Delta \to 0$ and shrink the segment to the point x, the acceleration of the center of mass of the segment becomes the same as the acceleration of the point at location x. This acceleration then becomes the second partial time derivative of the y-location of the segment at location x, $a_{\,\text{c of m},y} \to \partial^2 y(x,t)/\partial t^2$. Combining these limiting results we have

$$\frac{\partial T_y}{\partial x} = \mu \frac{\partial^2 y(x,t)}{\partial t^2}, \tag{8}$$

which makes the physical statement that for each tiny segment, it is the difference in the y-components of the tension which generates the acceleration.

There is a second, quite physical way to view (8). The net force on a system gives the time rate of change of its momentum. Because of Newton's third law of equal but opposite reaction forces, force gives the flow of momentum from one part of a system to another. The derivative $(\partial T_y/\partial x)\Delta$ thus gives the difference between the momentum flowing into a small chunk and the momentum flowing out. Thus should equal the time rate of change of the momentum of the chunk $mv = (\mu\Delta)\partial y/\partial t = p\Delta$, where we have defined the momentum per unit length (*momentum density*) as

$$p = \mu \frac{\partial y(x,t)}{\partial t}. \tag{9}$$

This quantity then allows us to rewrite (8) as

$$\frac{\partial T_y}{\partial x} = \mu \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t}\left(\mu \frac{\partial y(x,t)}{\partial t}\right) \\
&\Rightarrow \\
\frac{\partial T_y}{\partial x} &= \frac{\partial p}{\partial t}. \qquad (10)
\end{aligned}
$$

Turning back to the problem of finding a valid equation of motion, we must express *all* quantities in terms of specified constants and the solution $y(x,t)$. The one quantity remaining in (8) not yet in this form is the y-component of the tension T_y. To determine this component in terms of known quantities, we relate T_y to the known value of $T_x = \tau$ and the tangent of the angle of the line of action of the tension relative to the horizontal (θ in Figure 2. The slope of the string at any instant in time $\partial y(x,t)/\partial x$ gives the tangent of this angle. Thus,

$$
\begin{aligned}
\frac{T_y}{T_x \ (= \tau)} &= \tan\theta = \frac{\partial y}{\partial x} \\
&\Rightarrow \\
T_y &= \tau \frac{\partial y}{\partial x}. \qquad (11)
\end{aligned}
$$

An equation such as this which relates the basic driving forces in a system (T_y in this case) to the degrees of freedom is known generally as the *constitutive* relation.

Finally, substituting the result (11) for the tension into the y-component of Newton's law (8), we have the equation of motion,

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\tau \frac{\partial y}{\partial x}\right) &= \mu \frac{\partial^2 y}{\partial t^2} \\
&\Rightarrow \\
\frac{\partial^2 y}{\partial t^2} &= \frac{\tau}{\mu}\frac{\partial^2 y}{\partial x^2}.\text{E of M} \qquad (12)
\end{aligned}
$$

The equation of motion which we have found for the string (12) is precisely of the form of the famous *wave equation*,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad (13)$$

where c^2 is some positive constant, which in the case of waves on a string has the value

$$c = \sqrt{\frac{\tau}{\mu}} \quad \text{for strings.} \qquad (14)$$

The next section explores some solutions for the wave equation and determines the meaning of the mysterious constant c appearing in the wave equation.

4.3 Find a general solution

Finding a general solution to a partial differential equation such as (13) is a difficult task. As a prelude to this, we begin by considering examples of solutions which demonstrate important wave phenomena. As our first example, we consider a special class of solutions known as standing waves or normal modes.

4.3.1 Standing waves/Normal modes

As physical inspiration for writing down guess solutions to (13), we recall the demonstration from lecture in which by driving the string at certain frequencies, one creates very simple motions in which all segments of the string vibrate up and down perfectly in phase and with the same frequency.

To convert the observation of these natural motions into mathematical form, we consider the motion of the individual string segments. During the demonstration, a segment such as that labeled x_0 in Figure 1 exhibited simple harmonic motion in the y-direction. In constructing our coordinate system, we chose the

x-axis to lay along the string while it is at rest. Therefore, the equilibrium position of the simple harmonic motion for each segment is $y_{eq} = 0$. Hence, the motion of the particle at point x_0 should have the form

$$y_{x_0}(t) = A_{x_0} \cos(\omega t + \phi_0), \tag{15}$$

where A_{x_0} is the amplitude of the motion and ω and ϕ_0 are the angular frequency and initial phase, respectively. As noted above, we observed that all points move up and down with the same frequency and phase, with only the amplitude A_{x_0} depending upon the particular segment we are observing. Thus, for any point x (not just x_0, we may write

$$y_x(t) = A_x \cos(\omega t + \phi_0), \tag{16}$$

with the same angular frequency and phase for all points x. Mathematically, $y_x(t)$ gives the y-location for any value of x and t and so actually represents a two-variable function $y(x,t)$ and thus a particular type of solution to the wave equation. Converting (16) into more standard mathematical notation, we have the mathematical definition of a *standing wave* or, equivalently, a *normal mode*, as any solution of the form

$$y(x,t) = A(x) \cos(\omega t + \phi_0) \quad \text{math. def. of standing wave.} \tag{17}$$

To complete the specification of such as wave we must find appropriate functions $A(x)$ so that (17) satisfies the wave equation (13). To do this, we substitute (17) into the wave equation. First, we evaluate the appropriate derivatives,

$$\begin{aligned} \frac{\partial^2 y(x,t)}{\partial x^2} &= \frac{d^2 A(x)}{dx^2} \cos(\omega t + \phi_0) \\ \frac{\partial^2 y(x,t)}{\partial t^2} &= -\omega^2 A(x) \cos(\omega t + \phi_0). \end{aligned} \tag{18}$$

Here, we have used the facts that $A(x)$ depends only on x, so that x-derivatives of it are actually total derivatives and that two time derivatives of $\cos(\omega t + \phi_0)$ turn cos into $-\sin$ and $-\sin$ to $-\cos$ while pulling out two factors of ω. Next, substituting these results into (13), we find

$$\begin{aligned} c^2 \left(\frac{d^2 A(x)}{dx^2} \cos(\omega t + \phi_0) \right) &= \left(-\omega^2 A(x) \cos(\omega t + \phi_0) \right) \\ &\Rightarrow \\ \frac{d^2 A(x)}{dx^2} &= -\frac{\omega^2}{c^2} A(x). \end{aligned} \tag{19}$$

This latter equation looks just like the equation of motion of the simple harmonic oscillator, but with ω^2/c^2 instead of ω_0 and zero for the equilibrium position. We already know several forms of the general solution of such an equation. Thus, a general solution for $A(x)$ is

$$\begin{aligned} A(x) &= A_0 \cos\left(\frac{\omega}{c} x + \phi_1 \right) \\ &= A_0 \cos(kx + \phi_1) \quad k \equiv \omega/c \end{aligned} \tag{20}$$

where ϕ_1 is some initial phase, and we have written the cos in terms of x instead of t because x-derivatives appear in (19), and we identify the constant multiplying x as k in direct analogue to how we usually multiply t by ω. From this, we see that the angular frequency ω and the wave vector k describing the standing wave always come with the dispersion relation

$$\begin{aligned} k &= \frac{\omega}{c} \\ &\Rightarrow \\ \omega &= ck. \end{aligned} \tag{21}$$

From this, we at last determine the meaning of the constant c,

$$c = \frac{\omega}{k} = \frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} = \frac{\lambda}{T} = v.$$

Thus, in a standing wave, the ratio of the wavelength to the period is always the constant c, which for a string has the value $\sqrt{\tau/\mu}$. From (4), we see that this constant is precisely what we usually think of as the wave speed v.

Finally, given the general solution (20) for $A(x)$, we now have the general form for a standing wave,

$$y(x,t) = A_0 \cos(kx + \phi_1) \cos(\omega t + \phi_0). \tag{22}$$

4.3.2 Boundary Conditions

For any given wavelength, the relation (21) then determines the frequency. Thus, from what we have considered so far, all frequencies can lead to standing waves. The demonstration in lecture, however, shows that only certain frequencies result in standing wave solutions. Therefore, it appears that only certain wavelengths are allowed in the amplitude function $A(x)$ and, therefore, there must be additional conditions which we have not identified. These are known as boundary conditions.

What we have ignored is the motion of the very last segements of the string. The equation of motion derived from Figure 2 applies to interior segments *only.* Thus is because we implicitly assumed that each segment is in contact with additional segments to the left and to the right. The equations of motion for the segments at the end, or boundary, of the system will be different. These equations of motion are known as the *Boundary Conditions* (BC's).

There are as many different boundary conditions as there are things to which we can attach the ends of the string. To derive the boundary conditions, we proceed as with deriving any other equation of motion. We write down the laws of motion for the end of the string and express them entirely in terms of constants characterizing the system and the solution $y(x,t)$ and its derivatives. We now consider the two most most common types of boundary condition.

Fixed/Close Boundary Conditions — These conditions arrise from the type of boundary which we find in Figure 1. In this case, the ends of the string cannot move from the position $y = 0$ without breaking the string. (On the left the string knots around the attaching peg, and on the right the string feeds through the hole.) This type of boundary condition is thus termed a *fixed* boundary condition. Such a condition is also frequently termed *closed* because, in the case of sound, a closed end of a pipe prevents motion of the air and creates the same type of fixed boundary condition. Mathematically, a fixed boundary condition leads to the simple condition of zero displacement at the ends of the system,

$$y(x = x_0, t) = 0, \quad \text{(Fixed/Closed BC at location } x_0\text{)} \tag{23}$$

where x_0 gives the location of the fixed boundary ($x_0 = 0$ or L in Figure 1) and the condition holds for all times t.

Free/Open Boundary Conditions — This second boundary condition is more subtle. It arises when the end of the system is free to move. To realize this physically in the case of the string, we use the device in Figure 3a. Here, the end of the string attaches to a massless ring which is free to slide on a frictionless pole. We require the device of the ring to prevent the tension force $F = \tau$ from pulling the string through the hole in the wall at $x = L$. The ring, however, does not interfere with the motion of the string in the y-direction and thus leads to a *free* boundary condition. This condition is also termed *open* because, again in the case of sound, such a condition can be achieved in a pipe with an open end which allows the air to move freely in and out of the pipe.

To derive the mathematical form of this type of boundary condition, we again consider the motion of the end of the string. In this case, it attaches to the massless ring, and so to determine the laws of motion for this end, we consider the free-body diagram of the ring, as in Figure 3b. The only forces acting on the ring come from the contact with the frictionless pole and with the string. Because the pole is frictionless, the force from the pole is a pure normal force N. The force from the string is the tension force which, again, acts along the tangent direction to the string with components determined by (7,11), $T_x = \tau$ and $T_y = \tau \partial y(x_0,t)/\partial t$, where the partial derivative is evaluated at the location of the ring, which we here call

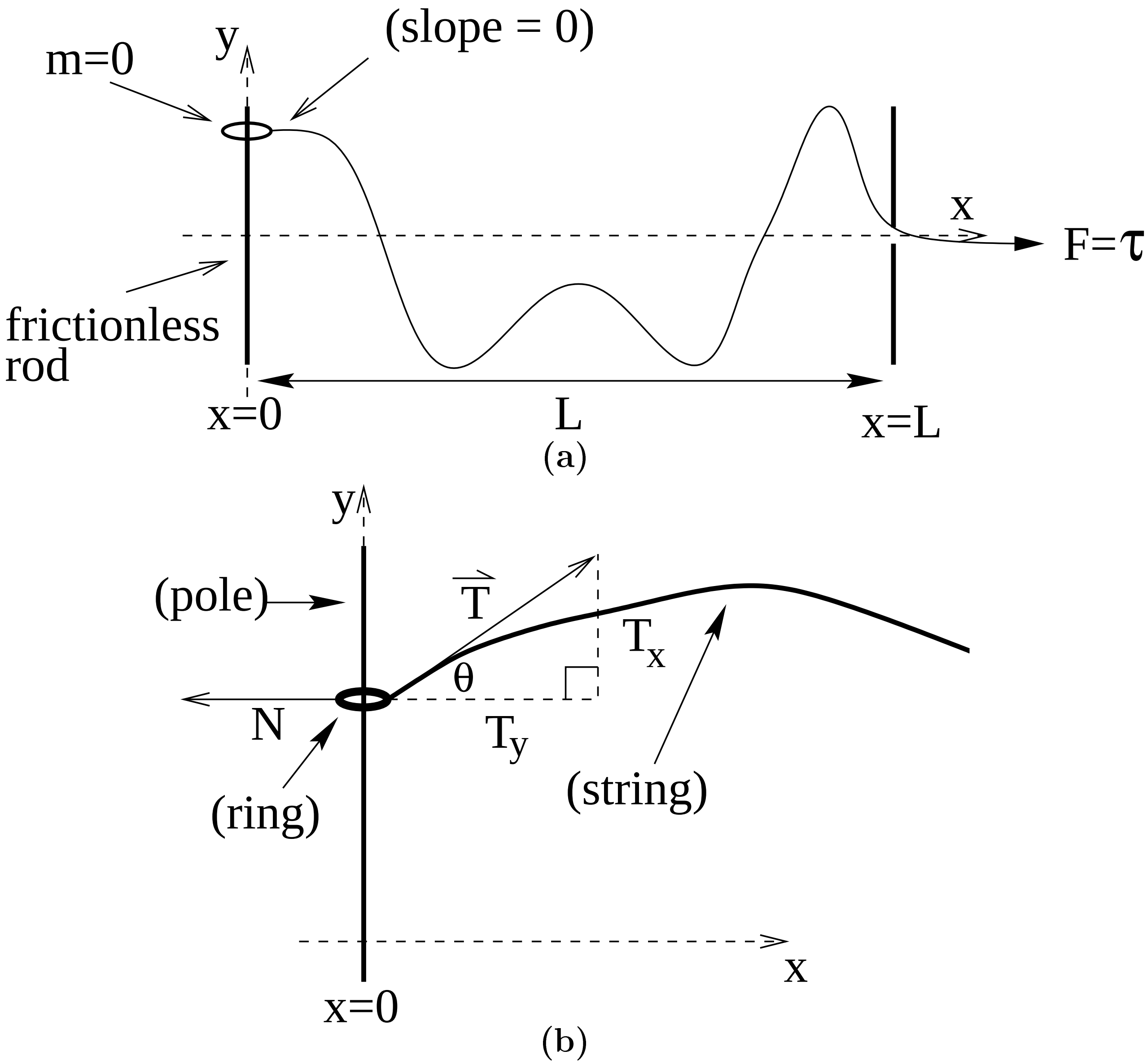

Figure 3: Free boundary condition: (a) physical realization; (b) free body diagram for ring

x_0. Because the ring has mass $m = 0$, we find for the x-component of Newton's law

$$\begin{aligned}
\sum F_x &= ma_x \\
-N + \tau &= 0 \cdot a_x \\
&\Rightarrow \\
N &= \tau,
\end{aligned}$$

and therefore conclude that the normal force acting on the ring is just force on the right with which we apply the tension to the string being transmitted along the full length of the string. From the y-component we learn,

$$\begin{aligned}
\sum F_y &= ma_y \\
\tau \frac{\partial y(x_0, t)}{\partial x} &= 0 \cdot a_y \\
&\Rightarrow \\
\frac{\partial y(x_0, t)}{\partial x} &= 0, \quad \text{(Free/Open BC at location } x_0\text{)}
\end{aligned} \tag{24}$$

that the solution must approach a free end with zero slope (as in the sketch in Figure 3a.) at all times t

5 What does it all mean?

In this set of notes, we have derived a number of results. It is useful not only to understand how these results arise directly from Newton's law, but to also interpret them physically so that we can better understand, remember and apply them. Below, we summarize the central results from these notes while interpreting them physically.

Degrees of Freedom $y(x)$ — A function $y(x)$ summarizes the degrees of freedom for the string. This is because we must give the y-location for each segment in order to know the configuration of the string. (We already know the x location for each segment because the segments do not move along x.) By sketching a function $y(x)$, we get a picture of the string and thus determine exactly what our system looks like at a given time, precisely what the degrees of freedom are supposed to do.

Solution $y(x, t)$ — We require a two-variable function to specify a solution for the string. This is because a solution should give the degrees of freedom at any time t. If we wish to know the degrees of freedom $y(x)$ at time t_0, we can get these from a function $y(x, t)$ by substituting the value of the time and generating a formula for $y(x)$: $y(x) = y(x, t = t_0)$.

Horizontal component of tension $T_x = \tau$ **(Eq. 7)** — The horizontal component of the tension remains constant across the entire string. The reason for this is that if the x-components of the tension were not constant, then some segments of the string would feel more tension on one side than on the other and this would cause them to move in the x-direction. As the segments of the string do not move in the x-direction, we conclude that the x-component of the tension must be constant throughout.

Vertical component of tension $T_y = \tau \partial y/\partial x$ **(Eq. 11,** *consitutive relation***)** — The y-component of the tension is in direct proportion to the tension set at the end of the string τ and to the slope of the string. For a given shape of the string, we naturally expect the tension to be in direct proportion to the tension τ applied to the string. The basic reason for the proportionality to the slope of the string is that the tension acts along the direction of the string and thus would have no y-component when the string is perfectly flat and has zero slope. Only when the string develops some slope ($\partial y/\partial x \neq 0$) will the tension have a y-component. The greater the slope, the greater the proportion of the tension in the y-direction. Thus, we expect T_y to be directly proportional also to $\partial y/\partial x$.

Segment response $\partial T_y/\partial x = \mu\partial^2 y/\partial t^2$ (Eq. 8) — There are two ways of looking at this equation. First, it basically just states Newton's law, $\sum F_y = ma_y$: on each segment two tension forces pull along y in opposite directions (hence the derivative), μ measures the mass of the segment and $\partial^2 y/\partial t^2$ is the acceleration. Another way is to rewrite it as $\partial^2 y/\partial t^2 = (1/\mu)\partial T_y/\partial x$, which says that the acceleration of each segment is in direct proportion to the rate of change in the y-component of the tension and in inverse proportion to the mass per unit length μ. The inverse proportion to μ comes directly from Newton's law $F = ma$. The smaller the mass of each segment the more acceleration $\partial^2 y/\partial t^2$ we expect. The proportionality to the derivative $\partial T_y/\partial x$ comes from the fact that tension pulls in opposite directions on the ends of each segment (Figure 2). If the tension in the y-direction were constant, then the two tensions would cancel and the segment would not accelerate. The greater the rate of change of the tension $\partial T_y/\partial x$, the greater the difference in tension on the two sides of a segment and the greater its acceleration. Thus, we do expect the acceleration $\partial^2 y/\partial t^2$ to be in direct proportion to $\partial T_y/\partial x$.

Conservation of momentum $\partial T_y/\partial x = \partial p/\partial t$ (Eq. 10) — The difference (per unit length) of the force acting on the two sides of a small chunk gives the time rate of change of the momentum per unit length of the chunk. This expresses conservation of momentum because it states that the net flow of momentum into the chunk gives the change of momentum in the chunk. This is a third way of looking at the segment response equation of the previous paragraph.

Equation of motion/wave equation: $\tau\partial^2 y/\partial x^2 = \mu\partial^2 y/\partial t^2$ (Eq. 12) — This equation comes directly from inserting our result for the y-component of the tension into the result for the segment response. To understand it physically, we can rearrange it as $\partial^2 y/\partial t^2 = (\tau/\mu)\partial^2 y/\partial x^2$. This states that the acceleration of each segment is directly proportional to the tautness of the string τ (which makes sense because there should be more acceleration when there are stronger tension forces), is inversely proportional to the mass density μ of the string (again, sensible because with more mass density, each segment is heavier and accelerates less easily) and is directly proportional to the curvature in the string. The reason why the amount of curvature determines the acceleration is that if there is no curvature (i.e., the string is straight), then the y-components of the tension on either end will be the same and will cancel, leaving no net force. The greater the curvature, the greater the difference in the y-components on either end of a segment, and thus the greater the acceleration.

Wave equation: $\partial^2 y/\partial t^2 = c^2\partial^2 y/\partial x^2$ (Eq. 13) — This is actually the standard form of a mathematical equation known as the wave equation. It is mostly a mathematical definition of a standard form for rewriting the equation of motion which we found. The most interesting part of this equation is to notice that c has the value $\sqrt{\tau}\mu$ for the string.

Dispersion relation: $\omega = ck$ (Eq. 21) — The frequency and wave-vector are in direct proportion through the constant c. To understand this, we consider a typical standing wave $y(x,t) = A_0\cos(kx)\cos(\omega t)$. Inserting into the wave equation, each spatial derivative gives a factor of k and each time derivative gives a factor of ω. Thus, we'll find $\omega^2 = c^2k^2$, and so $\omega = ck$. We also understand that c, therefore, is the wave-speed because then $c = \omega/k = (2\pi/T)/(2\pi/\lambda) = \lambda/T$.

Wave speed: $v = c = \sqrt{\tau/\mu}$ (Eq. 14) — The speed at which waves travel down the string (defined presently as the ratio of the wavelength λ to the period T of a standing wave) has the value $v = \sqrt{\tau/\mu}$. We find this result by just rearranging the equation of motion for the string into the standard wave equation form. This result for the wave speed has a very interesting similarity with the equation for the natural frequency of a harmonic oscillator $\omega_0 = \sqrt{k/m}$. In both cases the quantity which measures how quickly things happen in the system (the wave speed v for the string and the natural frequency ω_0 for the oscillator) being equal to the square-root of the ratio between something measuring the tightness of the restoring forces in the system (the tension τ in the case of the string and the spring constant k in the case of the oscillator) and something else measuring the inertia in the system (the mass per unit length μ for the string and the mass m for the oscillator). In both cases, it makes sense that the strength of the restoring forces appears on top (so that stronger forces lead to quicker responses) and that the inertia appears on the bottom (so that heavier systems respond more slowly).

Fixed/closed boundary condition: $y(x_0, t) = 0$ **(Eq. 23)** — If an end of the string at location $x = x_0$ is fixed, then the string at that point cannot move from having y-coordinate zero at any time t without breaking.

Free/open boundary condition: $\partial y(x_0, t)/\partial x = 0$ **(Eq. 24)** — If an end of the string at location $x = x_0$ is completely free to move in the y-direction, then the solution must approach that point with zero slope at all times t. Physically, we can understand this because we can imagine the free end of the string to be tied to a *massless* ring tied to a *frictionless* rod. Because the ring is massless and because of Newton's law $F = ma$, there cannot be any force on the ring in the y-direction. Any slope in the string, however, would immediately create a tension force on the ring in the y-direction according to our result for the vertical components of the tension. Thus, the solution $y(x, t)$ must always approach a free-end with zero slope.