Class Notes III: Other types of waves

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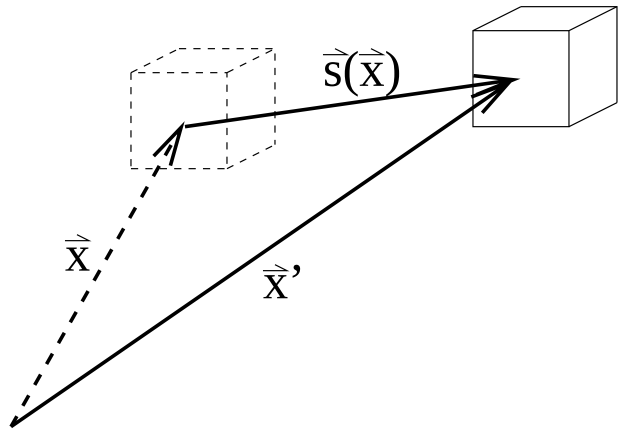

Figure 1: Mathematical description of disturbance in a sound wave: initial position of chunk with system at rest ($\vec{x}$), final position of chunk ($\vec{x}'$), displacement of chunk initially located at $\vec{x}$ ($\vec{s}(\vec{x})$).

1 Introduction

To underscore the ubiquity of wave behavior, we now consider two systems, quite different than the string under tension, whose equation of motion is also the wave equation. We shall see that not only the final result but also many of the equations leading up to the wave equation are analogous in all three systems.

2 Sound

Sound is nothing other than pressure waves. Not only gases but also fluids and solids can be put under pressure. Therefore, sound, occurs in all of these systems. We carry out the analysis below in such a way that it applies to all of these systems.

2.1 Degrees of freedom

The passage of a wave through any of the above systems disturbs, or moves, all of the particles making up the gas, fluid or solid. As with the string, the *degrees of freedom* must give some way to find the locations of each chunk making up the system. For the case of sound, we shall describe the disturbance by a vector $\vec{s}(\vec{x})$ which gives the *displacement* of the chunk which started at location $\vec{x}$ when the system was at rest. Because $\vec{s}(\vec{x})$ gives the displacement, if we wish to know the new location $\vec{x}'$ of the chunk which started at $\vec{x}$, we compute it by adding the displacement,

$$\vec{x}' = \vec{x} + \vec{s}(\vec{x}).$$

(See Figure 1.)

A *solution* for the sound wave, then would be a function $\vec{s}(\vec{x}, t)$ giving the displacement of each chunk for all times t. The location of each chunk at any time t is then

$$\vec{x}'(t) = \vec{x} + \vec{s}(\vec{x}, t), \tag{1}$$

The velocity and acceleration of the chunk which starts at location $\vec{x}$ are, respectively,

$$\vec{v}(\vec{x}, t) \equiv \frac{\partial \vec{x}'}{\partial t} = \frac{\partial \vec{x}}{\partial t} + \frac{\partial \vec{s}(x, t)}{\partial t} = \frac{\partial \vec{s}(x, t)}{\partial t} \tag{2}$$

$$\vec{a}(\vec{x}, t) \equiv \frac{\partial \vec{v}}{\partial t} = \frac{\partial^2 \vec{s}(\vec{x}, t)}{\partial t^2}, \tag{3}$$

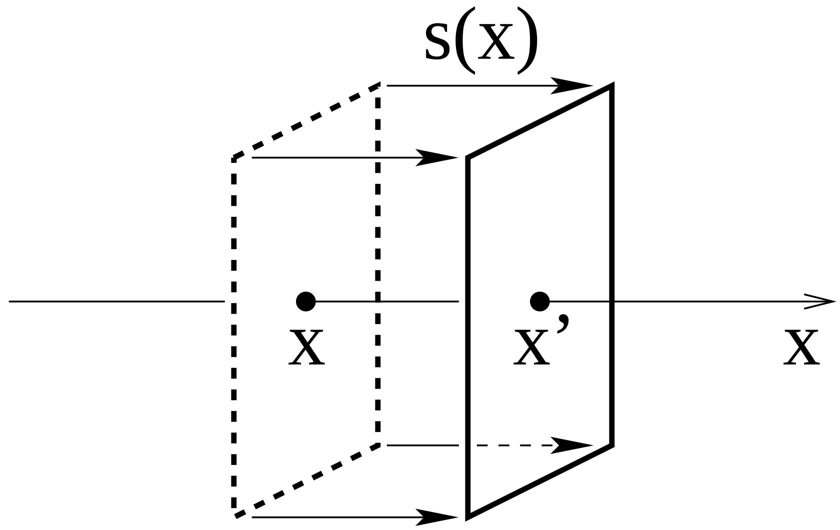

Figure 2: Plane wave with longitudinal polarization: direction of wave propagation (x), set of chunks undergoing the same displacement $s(x)$ (plane), initial location of chunks (dashed plane).

where we have used the fact that, by the very definition of partial derivative, $\partial\vec{x}/\partial t \equiv 0$.

2.2 Plane waves and polarization

Two facts allow us to dramatically simplify our analysis.

A *plane wave* is a wave in which the disturbance is constant in planes perpendicular to the direction of motion of the wave, which we shall always take to be along the x axis in these notes. (Figure 2 illustrates such a wave.) Any type of wave can be decomposed into a *superposition*, or sum, of plane waves. Hence, in these notes we consider the simpler case of plane waves without any loss of generality.

Polarization defines the direction of the motion associated with the disturbance relative to the direction of motion of the wave. In a string, for example, the direction of the motion associated with the wave is up-and-down along the y-axis, whereas the direction of motion of the wave is along the x-axis. These two directions are perpendicular, a special situation defined as *transverse polarization*. A pressure wave, on the other hand, requires expansion and contraction of the gas (or liquid or solid) to create changes in pressure. The gas compresses and expands only when the planes move back-and-forth along the x-axis. In this case, the direction of motion of the disturbance and of the propagation of the wave are parallel, a special situation defined as *longitudinal polarization*.

Thus, to study sound, we only need consider longitudinal plane waves. This means, as in Figure 2, that the displacement vector $\vec{s}(\vec{x})$ describes motion directly along the x axis. Thus, $\vec{s}$ only has a single component along the the x axis and can be described by a single scalar value s. Moreover, because the wave is plane, $s(\vec{x}) = s(x, y, z)$ is the same for all points which shares the same value of x, regardless of the values of y and z. Thus, a single scalar (non-vector) function $s(x)$ of the scalar x suffices to specify the degrees of freedom.

2.3 Equation of motion

2.3.1 Law of motion

To derive the equation of motion, we begin by writing Newton's law for the chunk of gas (or liquid or solid) sketched in Figure 3. As with the string, we ignore long-range forces as either irrelevant (electromagnetic) or insignificant (gravity). The only contact forces come from neighboring chunks. We describe this contact force

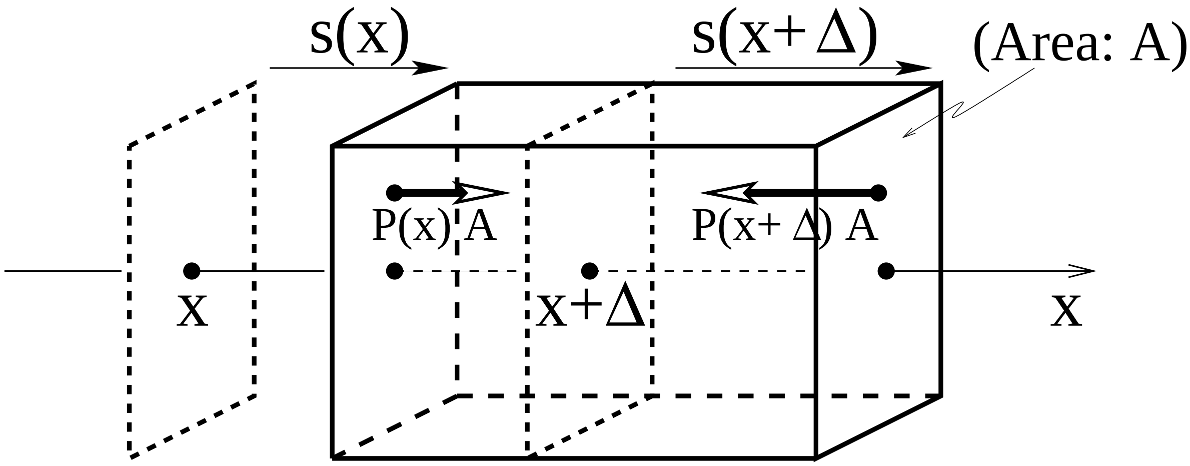

Figure 3: Free body diagram for pressure wave: present (solid planes) and initial (dashed planes) boundaries for chunk.

through the *pressure* P, defined as the force per unit area, acting across each face of the chunk. Pressure defines how much the neighboring chunks *push* on the chunk we have under consideration. Thus, as the figure indicates, the pressure force on the right-hand face points to the left whereas the pressure force on the left-hand faces points to the right. Finally, the passage of sound will change the pressure in the gas, and so we shall use P_0 to denote the pressure of the gas in the absence of any disturbance.

To describe the mass associated with the gas we shall use the *density* ρ, defined as the mass per unit volume, and we shall define ρ_0 as the density in the absence of any disturbance. To determine the mass m_{ch} of the chunk in Figure 3, we exploit conservation of mass and consider the mass of the chunk in its initial configuration in the absence of any disturbance, where it has density ρ_0, cross-sectional area A, and width Δ. Thus,

$$\begin{aligned} m_{ch} &= \rho_0 V_0 \\ &= \rho_0 \Delta \cdot A. \end{aligned} \quad (4)$$

Focusing on motion in the x-direction (corresponding to our degrees of freedom) and using Newton's law for finite bodies, we find

$$\begin{aligned} \sum F_{\text{ext},x} &= m_{ch} a_{\text{c of m},x} \\ -P(x+\Delta)A + P(x)A &= (\rho_0 \Delta \cdot A)\, a_{\text{c of m},x}, \\ -\frac{P(x+\Delta) - P(x)}{\Delta} &= \rho_0 a_{\text{c of m},x}, \end{aligned}$$

where in the last step we have canceled the common factors A and divided through by Δ. Finally, in the limit $\Delta \to 0$, the left-hand side becomes the partial derivative $\partial P/\partial x$, and the chunk shrinks down to the point x so that its acceleration becomes the same as the acceleration at the point x: $a_{\text{c of m},x} \to \partial^2 s/\partial t^2$ (Eq. 3). Combining these two results for the limit, we have

$$-\frac{\partial P}{\partial x} = \rho_0 \frac{\partial^2 s}{\partial t^2} \quad \text{(String: } \partial T_y/\partial x = \mu \partial^2 y/\partial t^2\text{)}, \quad (5)$$

which gives the law of motion for the chunk.

Note the similarity of (5) to the corresponding equation for the string. We have the first partial spatial derivative of the driving force equal to the product of a measure of the inertia and the acceleration of each chunk. The derivative represents the sum of the forces, which are in opposite directions on either side of the chunk and thus appear as a difference. The product of the measure of the inertia and the acceleration of each chunk represents the *ma* side of Newton's equation and ensures that the chunk's acceleration is directly proportional to the net driving force and inversely proportional to the inertia. The difference in signs of the left-hand sides between the sound and string laws of motion comes simply from the fact that pressures always push whereas tensions always pull. Otherwise, the equations are completely analogous.

Finally, following our analysis of the string, it can also be useful to consider the momentum density, in this case the momentum per unit *volume*. This momentum density is the product of the mass per unit volume ρ_0 and the velocity, $v = \partial s/\partial t$,

$$p = \rho_0 \frac{\partial s(x,t)}{\partial t}. \tag{6}$$

We can then rewrite (5) as relating the net flow of momentum into each chunk to the time rate of change of its momentum,

$$-\frac{\partial P}{\partial x} = \frac{\partial}{\partial t}\left(\rho_0 \frac{\partial s}{\partial t}\right) = \frac{\partial p}{\partial t} \quad \text{(String: } \partial T_y/\partial x = \partial p/\partial t). \tag{7}$$

2.3.2 Constitutive relation

Eq. 5 fails as an equation of motion only on its left-hand side where the pressure P appears rather than an explicit expression in terms of the degrees of freedom and their derivatives. To complete the derivation of the equation of motion, we must therefore determine the pressure in terms of the degrees of freedom. Such an explicit equation relating the driving forces in a system to the degrees of freedom is known as a *constitutive relation.*

To derive the constitutive relation for a gas (or fluid or solid), we begin by noting that as we increase the volume of the gas, we expect the pressure to decrease. For a small change in volume ΔV, we expect the change in pressure ΔP to also be small and in proportion to the change in volume, $\Delta P \propto -\Delta V$. Moreover, for a given change in volume ΔV, we expect the change in pressure to be quite small if the initial volume V_0 of gas is large and thus $\Delta P \propto 1/V_0$. Thus, we expect

$$\Delta P \equiv P - P_0 = -B\frac{\Delta V}{V_0}, \tag{8}$$

where B, the *bulk modulus*, is a constant characteristic of the particular material making up the system under study. (Note that throughout these notes, we always define our signs so that the change in quantity Q as $\Delta Q = Q - Q_0$ where Q_0 is the initial value of the quantity.)

To express the pressure in terms of the solution using (8), we begin by evaluating the relevant quantities for the chunk in Figure 3 directly in terms of the solution,

$$\begin{aligned} V_0 &= A\left\{(x+\Delta) - (x)\right\} \\ &= A\Delta \end{aligned} \tag{9}$$

$$\begin{aligned} \Delta V &= V - V_0 \\ &= A\left\{(x+\Delta+s(x+\Delta)) - (x+s(x))\right\} - A\Delta \\ &= A\left\{s(x+\Delta) - s(x)\right\}. \end{aligned} \tag{10}$$

Substituting these results into (8), we find the final constitutive relation relating the pressure P to the solution s,

$$\begin{aligned} P &= P_0 - B\frac{\Delta V}{V_0} \\ &= P_0 - B\frac{A\left\{s(x+\Delta) - s(x)\right\}}{A\Delta} \\ &= P_0 - B\frac{s(x+\Delta) - s(x)}{\Delta} \\ (\lim_{\Delta\to 0}) &\Rightarrow \\ P(x) &= P_0 - B\frac{\partial s}{\partial x} \quad \text{(String: } T_y = \tau\partial y/\partial x), \end{aligned} \tag{11}$$

where we have taken the limit of a very thin chunk, $\Delta \to 0$ in order to get the pressure at precisely the point x. Note again the similarity to the string. Apart from the constant background pressure P_0, which cancels out in most physical effects, the driving force is in direct proportion to the first spatial derivative of the solution through a constant characterizing the strength of restoring forces in the system, B for the gas (or liquid or solid) and τ for the string.

2.3.3 Final equation of motion

Substituting the constitutive relation (11) into the law of motion (5) we find the equation of motion for sound,

$$\begin{aligned} -\frac{\partial}{\partial x}\left(P_0 - B\frac{\partial s}{\partial x}\right) &= \rho_0 \frac{\partial^2 s}{\partial t^2} \\ &\Rightarrow \\ B\frac{\partial^2 s}{\partial x^2} &= \rho_0 \frac{\partial^2 s}{\partial t^2} \\ &\Rightarrow \\ \frac{\partial^2 s}{\partial t^2} &= \frac{B}{\rho_0}\frac{\partial^2 s}{\partial x^2} = c^2 \frac{\partial^2 s}{\partial x^2}. \quad \text{(String: } \partial^2 y/\partial t^2 = (\tau/\mu)(\partial^2 y/\partial x^2)\text{)} \end{aligned} \tag{12}$$

Thus, we see that sound obeys precisely the same wave equation as do waves on a string. In particular, we now have the wave speed of sound,

$$c = \sqrt{\frac{B}{\rho_0}}. \quad \text{(String: } c = \sqrt{\tau/\mu}\text{)} \tag{13}$$

As with waves on a string, we again find that the wave speed is the square root of the ratio of a measure of the restoring forces in the system to the inertia. Once again, a "stiffer" system (stronger restoring forces) leads to faster wave motion and increased inertia (mass density) leads to slower wave motion.

3 Electromagnetic Waves

We now investigate whether waves can propagate without *any* material system whatsoever, in vacuum. In particular, we consider electromagnetic (E&M) phenomena.

3.1 Degrees of freedom

The degrees of freedom of an electromagnetic system are the values of the electric and magnetic fields at all points $\vec{x}$ in space, $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$. We therefore now deal with a much more complex system where we now have to specify two vector fields for each point in space.

Again, we make the simplification of a plane wave and take the direction of propagation to be x. This means that all points with the same value of x will have the same electric and magnetic fields, regardless of the values of y and z. Mathematically, $\vec{E}(\vec{x}) = \vec{E}(x,y,z) = \vec{E}(x)$ and $\vec{B}(\vec{x}) = \vec{B}(x,y,z) = \vec{B}(x)$. Thus, the degrees of freedom are the value of these two fields for each x,

$$\left.\begin{array}{c}\vec{E}(x)\\ \vec{B}(x)\end{array}\right\} \quad \text{(D of F).}$$

The solution, then, must be something which gives the value of the degrees of freedom at all times t and so has the appearance

$$\left.\begin{array}{c}\vec{E}(x,t)\\ \vec{B}(x,t)\end{array}\right\} \quad \text{(solution).}$$

3.2 Equation of motion

The equation of motion expresses the basic physical laws governing the system in terms of the solution, its derivatives and constants describing the system.

3.2.1 Maxwell's Equations

We begin with the basic physical laws, which in this case are Maxwell's equations,

$$\begin{aligned} \oint \epsilon \vec{E} \cdot d\vec{A} &= Q^{\text{free}}_{\text{encl}} \\ &= 0 \end{aligned} \tag{14}$$

$$\begin{aligned} \oint \frac{1}{\mu} \vec{B} \cdot d\vec{\ell} &= I^{\text{free}}_{\text{encl}} + \frac{d}{dt} \int \epsilon \vec{E} \cdot d\vec{A} \\ &= \frac{d}{dt} \int \epsilon \vec{E} \cdot d\vec{A} \end{aligned} \tag{15}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{16}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}. \tag{17}$$

In the above equations, we have done two things a little differently than in your first class in E&M. First, we have written the equations in the proper form so that we can describe electromagnetic fields not only in vacuum but also inside of a material. In vacuum, ϵ and μ have the standard values

$$\begin{aligned} \epsilon_0 &= \frac{1}{4\pi k} = \frac{1}{4\pi \left(9 \cdot 10^9 \ \mathbf{J} \cdot \mathbf{m/C}^2\right)} \\ \mu_0 &= 4\pi \times 10^{-7} \ \mathbf{kg} \cdot \mathbf{m/C}^2. \end{aligned} \tag{18}$$

In a material, the values for these constants depend upon material and appear in reference books. Second, in vacuum Q and I refer to total enclosed charge and current, whereas inside a material Q and I refer to any excess, or "free", charges which are not a part of the material. Note that we have set these quantities to zero in both cases so that we are considering either fields in completely empty space or fields in a material where the only charges and currents are those which make up the material.

As a final comment, there is a standard notation for the combinations quantities appearing in the equation above,

$$\epsilon \vec{E} \equiv \vec{D} \tag{19}$$

$$\frac{1}{\mu} \vec{B} \equiv \vec{H} \tag{20}$$

3.2.2 Polarization

We have not yet said anything of the polarization of E&M waves. This is because such a determination requires us to consider the governing physics, namely (14-17). In particular, Gauss's laws (14) and (16) determine the nature of the polarization of electromagnetic waves.

The Gaussian "pillbox" in Figure 4 allows us to apply these laws to a plane wave traveling along x. Considering first the electric field, Eq. 14 tells us

$$\begin{aligned} 0 &= \oint \vec{E} \cdot d\vec{A} \\ &= \int_{(\text{top})} \vec{E} \cdot d\vec{A} + \int_{(\text{bottom})} \vec{E} \cdot d\vec{A} \end{aligned}$$

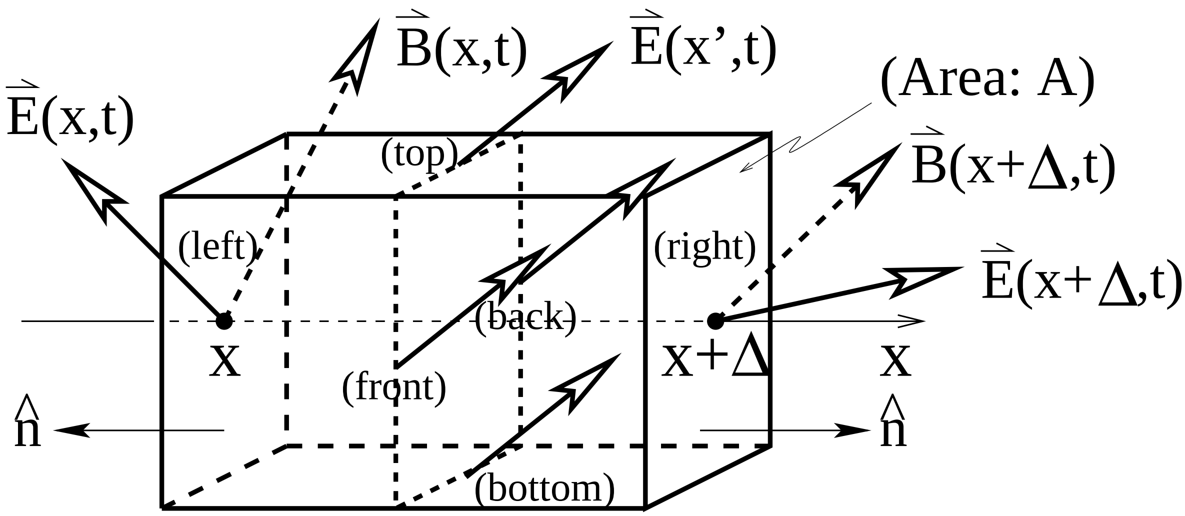

Figure 4: Pillbox for application of Gauss's laws to a plane wave traveling along the x direction.

$$+\int_{\text{(back)}} \vec{E}\cdot d\vec{A} + \int_{\text{(front)}} \vec{E}\cdot d\vec{A}$$
$$+\int_{\text{(left)}} \vec{E}\cdot d\vec{A} + \int_{\text{(right)}} \vec{E}\cdot d\vec{A},$$

where we have broken the closed surface integral around the pillbox into the sum of flux integrals over each of its faces. From the fact that we have a plane wave and thus that the electric field is constant in each plane (such as x' in the figure), we see that (a) the electric flux entering the bottom of the pillbox equals exactly the electric flux exiting the top and that (b) the flux entering the front equals exactly the flux exiting the back. Thus, the top and bottom integrals sum to zero and the back and front integrals sum to zero. This leaves

$$0 = \int_{\text{(left)}} \vec{E}\cdot d\vec{A} + \int_{\text{(right)}} \vec{E}\cdot d\vec{A}, \tag{21}$$

Due to the nature of the plane wave, we are fortunate in evaluating the remaining integrals over the left and right faces because the electric field is *constant* over each. In both cases, we take the dot product of the electric field with the outward normal vector to the surface $\hat{n}$ and multiply by the area A. As indicated in the figure, the normal vector to the left face is $-\hat{x}$ and to the right face is $+\hat{x}$. Substituting this into (21) and simplifying,

$$\begin{aligned} 0 &= \left(\vec{E}(x,t)\cdot(-\hat{x})\right)A + \left(\vec{E}(x+\Delta,t)\cdot(\hat{x})\right)A \\ &= A\left(-E_x(x,t) - E_x(x+\Delta)\right) \\ &\Rightarrow \\ E_x(x+\Delta) &= E_x(x), \end{aligned} \tag{22}$$

where we have used the fact that the components of any vector may be obtained by dotting with the corresponding unit vector: $v_x = \vec{v}\cdot\hat{x}$, $v_y = \vec{v}\cdot\hat{y}$, $v_z = \vec{v}\cdot\hat{z}$.

Eq. 22 tells us that the x-component of $\vec{E}$ is constant. To determine the value of this constant, we do as we did with the x-component of the tension in the string and consider the value of E_x at the edge of the system. Unlike the string, in this case our system has no actual end and extends off to infinity. Very far

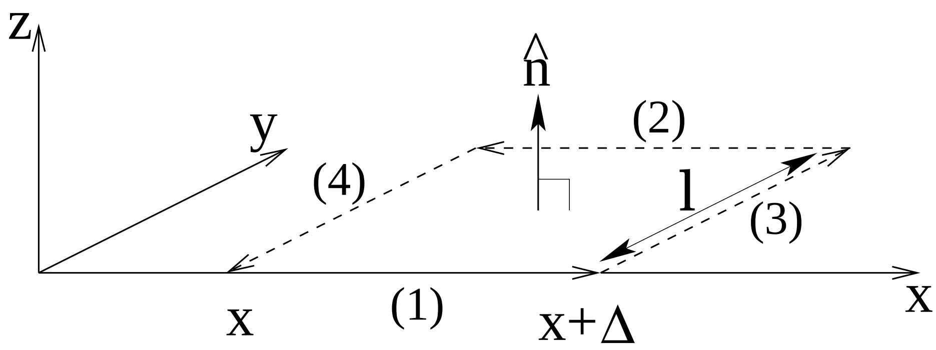

Figure 5: Amperean loop for application of Ampere's and Faraday's laws to a plane wave traveling along the x direction.

from the sources of any fields, we expect the fields $\vec{E}$ and $\vec{B}$ to be zero. Thus the value of the constant is zero and so

$$E_x(x,t) = 0 \quad \text{for all } x \text{ and } t. \tag{23}$$

The fact that the x component of $\vec{E}$ is zero is best summarized by saying that $\vec{E}$ is perpendicular to the direction of propagation. In other words, the electric field in an electromagnetic wave is transversely polarized.

Turning next to the magnetic field, we now apply (16). Note that (16) is *identical* to (14) but with $\vec{B}$ replacing $\vec{E}$. Thus, the analysis for the magnetic field will follow exactly the same logic and generate Eqs. 21-23 but with E replaced everywhere by B. Thus, we conclude

$$B_x(x,t) = 0 \quad \text{for all } x \text{ and } t, \tag{24}$$

and that *both* the electric *and* magnetic fields in an electromagnetic wave are *transversely polarized*.

3.2.3 Laws of motion

Having determined the polarization, we turn next to the equations describing laws of motion of the fields, (15) and (17), and express them in terms of the solution ($\vec{E}(x,t)$ and $\vec{B}(x,t)$), its derivatives and fundamental constants. Figure 5 shows the Amperean loop which we shall use to study these laws.

Applying Ampere's law (15) to this loop, we find

$$\begin{aligned} \oint \frac{1}{\mu}\vec{B}\cdot d\vec{\ell} &= \int \epsilon\vec{E}\cdot d\vec{A}, \qquad (25)\\ \int_{(1)} \frac{1}{\mu}\vec{B}\cdot d\vec{\ell} + \int_{(2)} \frac{1}{\mu}\vec{B}\cdot d\vec{\ell} &= \\ + \int_{(3)} \frac{1}{\mu}\vec{B}\cdot d\vec{\ell} + \int_{(4)} \frac{1}{\mu}\vec{B}\cdot d\vec{\ell} & \\ + \int_{(3)} \frac{1}{\mu}\vec{B}\cdot d\vec{\ell} + \int_{(4)} \frac{1}{\mu}\vec{B}\cdot d\vec{\ell}. & \end{aligned}$$

Here, we have broken the Amperean loop into the four segments in the figure and used the fact that the magnetic field is transversely polarized and thus $\vec{B}\cdot d\vec{\ell} = 0$ along sides (1) and (2). Again, we are fortunate in that along each of the contributing integrals, sides (3) and (4), the field $\vec{B}$ is constant. Thus, for these integrals, we have just the dot product of the value of $\vec{B}$ with the corresponding direction times the length ℓ of the side. As sides (3) and (4) are along $+\hat{y}$ and $-\hat{y}$, the dot products pick out the y component of the field $\pm B_y$. As for the right-hand side of (25), $d\vec{A}$ is oriented perpendicular to the loop according to the right-hand rule and thus is along the direction $\hat{n} = +\hat{z}$ in the figure. Finally, the total area of the loop is $A = \ell\Delta$. Putting this all together, we find

$$\begin{aligned} \frac{1}{\mu}B_y(x+\Delta,t)\ell - \frac{1}{\mu}B_y(x,t)\ell &= \frac{d}{dt}\left(\langle\epsilon E_z\rangle\ell\Delta\right) \\ &\Rightarrow \\ \frac{\frac{1}{\mu}B_y(x+\Delta,t) - \frac{1}{\mu}B_y(x,t)}{\Delta} &= \frac{d}{dt}\langle\epsilon E_z\rangle \end{aligned} \tag{26}$$

Here, the only subtlety is that the value of E_z is not constant across the face of the loop. Thus, what we find for the integral is the *average* value of E_z times the area of the loop. This is much akin to what we found for the string and for sound where we end up with an equation involving the location of the center of mass of a chunk of the system. The resolution of this, as in the other two cases, is to take the limit $\Delta \to 0$ so that the loop shrinks down to the point x so that the average value just becomes the value at x at the particular point in time, $\langle E_z\rangle \to E_z(x,t)$. Taking the limit, Eq. 26 thus gives our first law of motion

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}B_y(x,t)\right) = \frac{\partial}{\partial t}\left(\epsilon E_z(x,t)\right) \tag{27}$$

Turning next to Faraday's law (17), we can repeat the same procedure. For this analysis, we note that Faraday's law (17) looks just like Ampere's law (15) but with $(1/\mu)\vec{B}$ replaced with $\vec{E}$ and with $\epsilon\vec{E}$ replaced with $-\vec{B}$. Thus, the derivation for Faraday's law will just repeat Eqs. 25-27 but with the aforementioned replacements at each and every step, leading to the final result

$$\frac{\partial}{\partial x}\left(E_y(x,t)\right) = -\frac{\partial}{\partial t}\left(B_z(x,t)\right) \tag{28}$$

As we have four unknown degrees of freedom (E_y, E_z, B_y, B_z — From the polarization we already know that $E_x = B_x = 0$.), we shall require four equations. The other two equations come from considering the loop in the xz plane shown in Figure 6. The analysis will follow exactly as before with (25) with the loop integrals along (1) and (2) being zero because the fields are transverse but with the integrals along (3) and (4) now picking out the z-components $\pm(1/\mu)B_z$. The area integral for $\vec{E}$ also works similarly, but now because $\hat{n}$ points along $-\hat{y}$, the right-hand side picks up an extra minus sign. Our final result looks like (27), but with these changes:

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}B_z(x,t)\right) = \frac{\partial}{\partial t}\left(-\epsilon E_y(x,t)\right). \tag{29}$$

For the final equation, we apply Faraday's law to the second loop. Making the same changes which we made above to generate Faraday's law from Ampere's law for the first loop, we find

$$\frac{\partial}{\partial x}\left(E_z(x,t)\right) = -\frac{\partial}{\partial t}\left(-B_y(x,t)\right) = \frac{\partial}{\partial t}\left(B_y(x,t)\right). \tag{30}$$

3.2.4 Analogies with sound and strings

Notice the similarity of the law of motion (27) to the law of motion for sound (7) and the corresponding equation string. We find the spatial derivative of one quantity to be related directly to the time derivative of another. It thus appears that $(1/\mu)B_y = H_y$ plays the role of $B\partial s/\partial x = -(P - P_0)$, some sort of driving force, and ϵE_z plays the role of $p = \rho_0\partial s/\partial t$, some sort of momentum. Carrying the analogy further, we thus expect that $1/\mu$ plays the role of B, some sort of stiffness, and that ϵ plays the role of ρ_0, some sort of inertia. If this analogy is to be complete, then we need also that B_y play the role of $\partial s/\partial x$, the spatial derivative of some quantity, and that E_z play the role of $\partial s/\partial t$, the time derivative of the *same* quantity.

In general, it is not always possible to define two quantities like B_y and E_z to arise as two different derivatives of a *single* quantity[1]. What allows us to do so in this case, and thus make the analogy complete,

[1] The material in this paragraph is included for your interest. You are not responsible for it "officially".

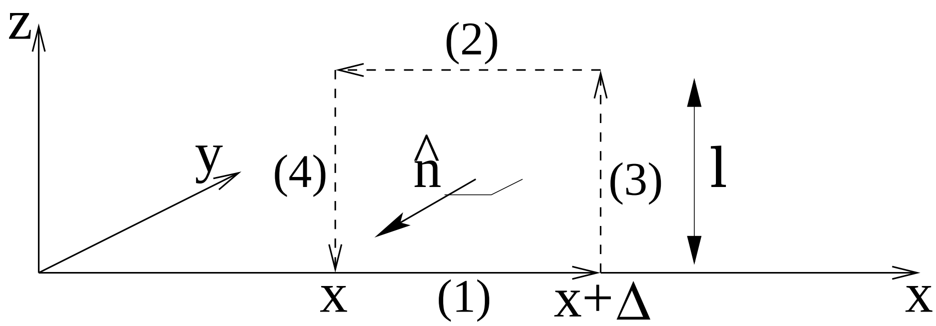

Figure 6: Second Amperean loop for application of Ampere's and Faraday's laws to a plane wave traveling along the x direction.

is Faraday's law (30). It is a theorem of partial differential calculus that the mixed partial derivatives of a single quantity ($\partial^s/(\partial x \partial t)$ and $\partial^s/(\partial t \partial x)$) are equal. Thus, if we define B_y to be the x derivative of some quantity and E_z to the time derivative of some quantity, then we must have that the t derivative of B_y equal the x derivative of E_z. A further theorem states that so long as these latter two derivatives indeed are equal (as Faraday's law ensures for us!), then we indeed always can define a single quantity whose derivatives yield the original two quantities. In the present case, this particular quantity has a name in advanced courses in electromagnetism, the *vector potential* $\vec{A}$. In such courses, you will learn that in fact

$$
\begin{aligned}
E_z &\equiv \frac{\partial}{\partial t}(-A_z) \\
B_y &\equiv \frac{\partial}{\partial x}(-A_z),
\end{aligned}
\tag{31}
$$

and thus what plays the role of s is exactly the quantity $-A_z$.[2] (The minus signs are a matter of convention.) As a final remark, if we now insert the definitions (31), which Faraday's law allows us to make, into Ampere's law (27), we could derive the wave equation,

$$
\frac{1}{\mu}\frac{\partial^2 A_z}{\partial x^2} = \epsilon \frac{\partial^2 A_z}{\partial t^2},
$$

and immediately obtain the wave speed for electromagnetic waves, $c = \sqrt{1/(\mu\epsilon)}$. As the *vector potential* is not part of the official material for this course, we take a somewhat more roundabout approach to deriving the wave equation in the next section.

3.2.5 Wave equation for the electric and magnetic fields

Now in possession of the laws of motion in differential form (27,28,29,30), we have only to see whether the components of the fields actually satisfy the wave equation. We begin with

$$
\begin{aligned}
\frac{\partial^2 E_y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial E_y}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(-\frac{1}{\mu\epsilon}\frac{\partial B_z}{\partial x}\right) \quad \text{; from (29)} \\
&= -\frac{1}{\mu\epsilon}\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) \quad \text{; switch order of derivs}
\end{aligned}
$$

[2] A similar quantity is traditionally defined for E_y and B_z so that $E_y = -\partial A_y/\partial t$ and $B_z = \partial A_y/\partial x$. To practice what you've learned in the above discussion, verify that you can explain the difference in minus sign for B_z.

$$
\begin{aligned}
&= -\frac{1}{\mu\epsilon}\frac{\partial}{\partial x}\left(-\frac{\partial E_y}{\partial x}\right) \quad ; \text{from (28)} \\
&= \frac{1}{\mu\epsilon}\frac{\partial^2 E_y}{\partial x^2} \\
&= c^2\frac{\partial^2 E_y}{\partial x^2} \quad ; \text{wave equation with } c \equiv \sqrt{\tfrac{1}{\mu\epsilon}}!!! \qquad (32)
\end{aligned}
$$

Thus, indeed, the y-component of the electric field satisfies the wave equation and exhibits all the same wave phenomena (such as standing waves) as do strings and sound. Using the values of the constants appropriate for vacuum (18), we find the speed of these waves to be

$$
\begin{aligned}
c &= \sqrt{\frac{1}{\mu_0\epsilon_0}} \\
&= \sqrt{\frac{1}{4\pi\cdot 10^{-7}\mathbf{kg\text{-}m/C^2}\frac{1}{4\pi(9\cdot 10^9)\mathbf{J\text{-}m/C^2}}}} \\
&= \sqrt{\left(\frac{(4\pi)9\cdot 10^9}{(4\pi)\cdot 10^{-7}}\right)\left(\frac{\mathbf{J\text{-}m/C^2}}{\mathbf{kg\text{-}m/C^2}}\right)} \\
&= \sqrt{(9\cdot 10^{16})\left(\frac{\mathbf{m}^2}{\mathbf{s}^2}\right)} \\
&= 3\cdot 10^8\mathbf{m/s},
\end{aligned}
$$

which is precisely the speed of light in vacuum!!! We can imagine what a day Maxwell must have had when he first obtained this result. This establishes that the light that we see actually may be understood as electromagnetic waves. Amazingly, the constants ϵ_0 and μ_0 of ordinary E&M laboratory experiments involving charges, currents and electric and magnetic fields are precisely what are needed to understand the seemingly completely different phenomenon of light!

Following the same steps that we did to derive (32), we can verify that all of the other components of the fields satisfy the wave equation. The key observation here is that they all obey the same wave equation *with the same wave speed*, so that the fields always move together. These derivations all follow the same general strategy of starting with the second time derivative, using one of the differential versions of Maxwell's equation, switching the order of the resulting mixed partial derivatives, and then using the differential versions of another Maxwell's equation.

$$
\begin{aligned}
\frac{\partial^2 B_y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial B_y}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial E_z}{\partial x}\right) \quad ; \text{from (30)} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial t}\right) \quad ; \text{switch order of derivs} \\
&= \frac{\partial}{\partial x}\left(\frac{1}{\mu\epsilon}\frac{\partial B_y}{\partial x}\right) \quad ; \text{from (27)} \\
&= \frac{1}{\mu\epsilon}\frac{\partial^2 B_y}{\partial x^2} \\
&= c^2\frac{\partial^2 B_y}{\partial x^2} \quad ; \text{wave equation with } c \equiv \sqrt{\tfrac{1}{\mu\epsilon}}!!! \qquad (33)
\end{aligned}
$$

$$
\frac{\partial^2 E_z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial E_z}{\partial t}\right)
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t}\left(\frac{1}{\mu\epsilon}\frac{\partial B_y}{\partial x}\right) \quad \text{; from (27)} \\
&= \frac{1}{\mu\epsilon}\frac{\partial}{\partial x}\left(\frac{\partial B_y}{\partial t}\right) \quad \text{; switch order of derivs} \\
&= \frac{1}{\mu\epsilon}\frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial x}\right) \quad \text{; from (30)} \\
&= \frac{1}{\mu\epsilon}\frac{\partial^2 E_z}{\partial x^2} \\
&= c^2\frac{\partial^2 E_z}{\partial x^2} \quad \text{; wave equation with } c \equiv \sqrt{\tfrac{1}{\mu\epsilon}}\text{!!!} \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 B_z}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(-\frac{\partial E_y}{\partial x}\right) \quad \text{; from (28)} \\
&= -\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial t}\right) \quad \text{; switch order of derivs} \\
&= -\frac{\partial}{\partial x}\left(-\frac{1}{\mu\epsilon}\frac{\partial B_z}{\partial x}\right) \quad \text{; from (29)} \\
&= \frac{1}{\mu\epsilon}\frac{\partial^2 B_z}{\partial x^2} \\
&= c^2\frac{\partial^2 B_z}{\partial x^2} \quad \text{; wave equation with } c \equiv \sqrt{\tfrac{1}{\mu\epsilon}}\text{!!!} \qquad (35)
\end{aligned}
$$

4 Summary of connections

We have found that the quite disparate systems of a string, pressure wave (in a gas, fluid or solid), and vacuum (or a material with internal electric and magnetic fields) all ultimately obey the same wave equation, $\partial^2 q/\partial t^2 = c^2\partial^2 q/\partial x^2$, where q is some quantity describing the degrees of freedom in the system.

We have found there to be deeper connections, however, which help to explain the ubiquity of the wave equation. On the string and in sound, we can define a quantity describing driving forces in the system in terms of the first spatial derivative of the solution and a quantity describing momentum in the system in terms of the first time derivative of the solution. We then find that the wave equation comes from relating the first spatial derivative of the driving forces (which gives the net force on a chunk) to the first time derivative of the momentum density, which is just Newton's law that total net force equals rate of change of momentum. For electromagnetics, there is no longer Newton's law but Maxwell's equations. Nonetheless, we find exactly analogous quantities and equations. Table 1 summaries the analogy.

Quantity Name	String	Sound	E&M
Dynamical Equation	$\frac{\partial}{\partial x}T_y = \frac{\partial}{\partial t}p$	$-\frac{\partial}{\partial x}P = \frac{\partial}{\partial t}p$	$\frac{\partial}{\partial x}H_y = \frac{\partial}{\partial t}D_z$
Driving Force	$T_y = \tau\frac{\partial}{\partial x}y$	$P = P_0 - B\frac{\partial}{\partial x}s$	$H_y = \frac{1}{\mu}B_y = -\frac{1}{\mu}\frac{\partial}{\partial x}A_z$
Stiffness	τ	B	$\frac{1}{\mu}$
Displacement	y	s	A_z
Momentum density	$p = \mu\frac{\partial}{\partial t}y$	$p = \rho_0\frac{\partial}{\partial t}s$	$D_z = \epsilon E_z = -\epsilon\frac{\partial}{\partial t}A_z$
Inertia density	μ	ρ_0	ϵ
Wave Equation	$\frac{\partial^2}{\partial t^2}y = c^2\frac{\partial^2}{\partial x^2}y$	$\frac{\partial^2}{\partial t^2}s = c^2\frac{\partial^2}{\partial x^2}s$	$\frac{\partial^2}{\partial t^2}(A_z, E_y, B_z,\ etc.) = c^2\frac{\partial^2}{\partial x^2}(A_z, E_y, B_z,\ etc.)$
Wave Speed	$\sqrt{\frac{\tau}{\mu}}$	$\sqrt{\frac{B}{\rho_0}}$	$\sqrt{\frac{1/\mu}{\epsilon}} = \sqrt{\frac{1}{\mu\epsilon}}$

Table 1: Physics analogy among strings, sound, and E&M systems