Class Notes I: "Simple" Harmonic Motion (SHM)

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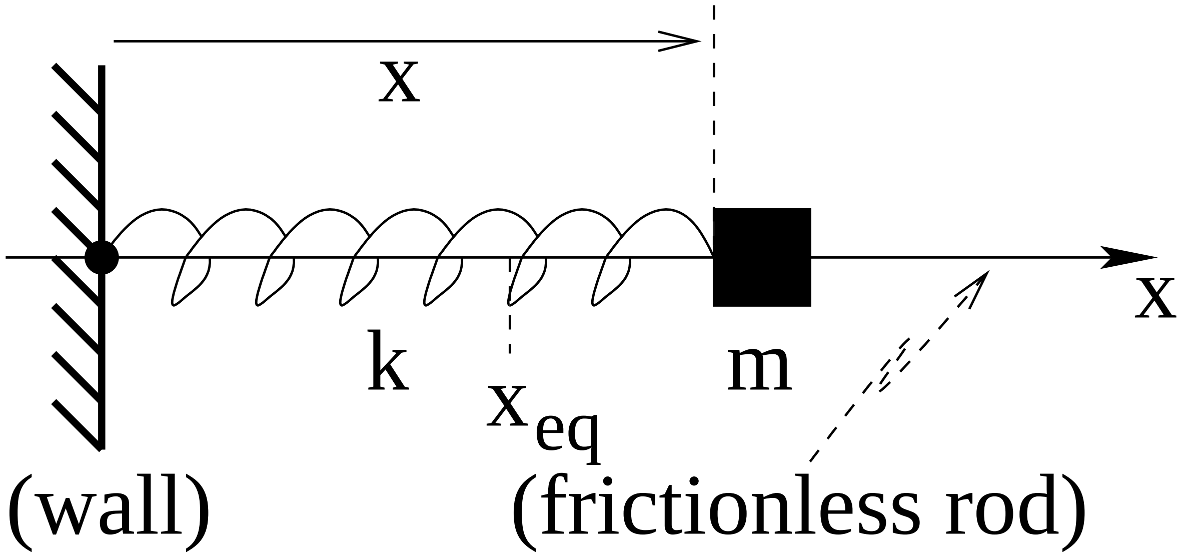

Figure 1: Simple harmonic oscillator (SHO): spring-mass system illustrating simple harmonic motion

1 Motivation

Simple harmonic motion (SHM) refers to a type of motion which repeats cyclically, or periodically, in time. Our motivation for study of such motion is threefold: (1) it is an important component of wave motion, (2) in reviewing the analysis of SHM we can develop a general physical plan of attack to address the more complex types of motion associated with waves (Section 4, (3) we will use the familiar results from the analysis of SHM to introduce the powerful technique of analysis using complex numbers (Section 5).

2 Physical realization

Harmonic behavior occurs in many types of systems, including both electronic and mechanical systems. The first step in analyzing a new phenomenon is to choose a particular system for study which exhibits that phenomenon. Our particular system is the *simple harmonic oscillator* (SHO), the spring-and-mass system in Figure 1. This system consists of a spring of constant k and equilibrium length x_{eq} attaching a mass m to a solid wall located at $x = 0$. To keep the problem as simple as possible, we consider motion in the x-direction only. To accomplish this, we may consider the mass to slide on a frictionless rod oriented in the x-direction. The motion of this system repeats periodically in time. Our goal is to characterize and analyze this behavior.

3 Basic Characterization

The second step in understanding a new phenomena is to identify the basic quantities which we hope to describe and understand. In the case of the SHO (Figure 1), we can think in terms of describing (1) the periodicity of the system and (2) the motion of the system.

3.1 Periodicity

There are three common ways of quantifying periodic behavior in time. (When we begin to look at wave motion, we shall develop directly analogous quantities to describe periodicity *in space*.):

- *Period* T: The time for a full period or cycle. The period is typically measured in seconds, so that the basic unit is 1 **sec**.
- *Frequency* f: The number of cycles which occur in a unit time. The unit of frequency is typically 1 **cycle/sec**, which is defined as a special unit, 1 **hertz** (1 **Hz**).
- *Angular frequency* ω: The number of radians of phase which pass in a unit time, where we associate 2π radians of phase with each cycle. The unit of angular frequency is thus typically **1 radian/sec**. Because radians carry no dimension (2π **radian** is the ratio of the circumference of a circle to the radius, both measured in meters thus resulting in canceling units.), we frequently omit the radian and write the unit of angular frequency as simply 1 $\mathbf{sec}^{-1}$.

All three of these measures describe basically the same thing and can be converted among each other. We shall use them interchangeably, and it thus is important that you be able to convert among them quickly using the following conversions.

If one cycle takes time T, then the number of cycles which occur per unit of time is

$$f = \frac{1\ \textbf{cycle}}{T}. \tag{1}$$

Because there are 2π radians in one cycle, the number of radians which pass per unit time are therefore

$$\omega = \frac{2\pi\ \textbf{rad}}{T} = \frac{2\pi}{T}, \tag{2}$$

where we remind ourselves that we may drop the optional "unit" **rad**. Finally, combining (1-2), we have

$$\omega = 2\pi f. \tag{3}$$

3.2 Motion

Figure 2 shows the motion typical of the SHO system, plotting the position of the mass $x(t)$ along the vertical axis as a function of time t along the horizontal axis. We shall use the following quantities to characterize this motion.

- *Amplitude* A: the maximum displacement from the equilibrium position $x_{\max} - x_{eq}$, also equal to one-half of the total range of motion $\frac{1}{2}(x_{\max} - x_{\min})$.
- *Phase of the oscillator at time* t $\phi(t)$: the time elapsed t_{el} since the most recent occurrence of the maximum displacement in the motion ($x(t) = x_{\max}$), measured in radians where we associate 2π **rad** of phase with each cycle. Thus, $\phi(t) = 2\pi t_{el}/T = \omega t_{el}$.
- *Initial phase* ϕ_0: the phase of the oscillator at time $t = 0$, $\phi_0 \equiv \phi(t = 0)$.

Note that with the above definitions, we can write down the motion immediately as

$$x(t) = x_{eq} + A\cos(\omega t + \phi_0). \tag{4}$$

4 General Analysis

After identifying our physical realization and defining our basic descriptive terms, the next step in attacking a new phenomena is analysis of the associated motion. This involves a series of general steps which we often take for granted in relatively simple systems such as the SHO. We aim here to make those steps explicit so that we will be prepared to take on the much more complex systems which we will have to study to understand wave behavior.

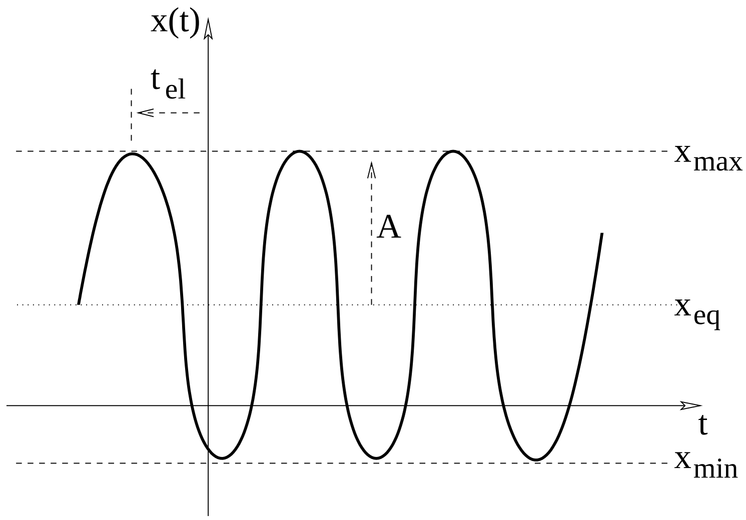

Figure 2: Simple harmonic motion: position $x(t)$ (solid curve), equilibrium position x_{eq} (dotted line), amplitude A (vertical dashed line), time elapsed since the most recent maximum t_{el} as appears in the definition of the initial phase ϕ_0 (horizontal dashed arrow).

4.1 Identify the degrees of freedom

The *degrees of freedom* (D of F) of a system are the minimal set of variables needed to define the configuration of the system at a given instant in time. Note that because we wish to describe only a given instant in time, quantities which involve changes in time, such as velocities, need not be specified.

In the example of the SHO (Figure 1), we only need one variable, the position of the mass x to define the configuration of the system at any given time t. There is thus only one degree of freedom in our present system, $x(t)$[1].

4.2 Derive the equation of motion

The *equation of motion* (E of M) of a system is an equation which expresses the fundamental laws of motion in terms of *only* (a) the degrees of freedom, (b) derivatives of the degrees of freedom, and (c) constants characterizing the system.

In the example of the SHO, the fundamental law of motion is Newton's law. Because our particle is free to move along the x-axis only, we need consider just the x-components of the motion. Newton's law states for this motion,

$$\sum F_x = ma_x.$$

The only force acting along the x-direction (because the rod is frictionless) comes from the spring and is in direct proportion to the stretch in the spring $x(t) - x_{eq}$ through the spring constant k. Moreover, when the spring is stretched, $x(t) > x_{eq}$ (and $x(t) - x_{eq} > 0$), the force is *backwards* along the x-axis. Therefore, $\sum F_x = -k(x(t) - x_{eq})$. Thus, we have

$$-k\left(x(t) - x_{eq}\right) = ma_x.$$

The above equation does not yet qualify as an equation of motion because the acceleration a_x does not fall into the allowed categories (a-c). We amend this simply by replacing the acceleration in the x-direction

[1]Note that one could also perhaps describe the configuration of the system by giving the amplitude A and the phase ϕ, but this requires two variables, not just one, and so is not a *minimal* set of variables.

by its mathematical equivalent $a_x = d^2x(t)/dt^2$ and, thereby, derive the equation of motion for the SHO,

$$-k(x(t) - x_{eq}) = m\frac{d^2x(t)}{dt^2} \quad \text{E of M} \tag{5}$$

Eq. (5) now qualifies as an equation of motion because all terms fall into the above categories: $x(t)$ is in (a), $d^2x(t)/dt^2$ is in (b), and $-k$, x_{eq} and m are in (c).

4.3 Find a general solution

A *solution* is an explicit mathematical formula for the values of the degrees of freedom as a function of time which satisfies the equation of motion *at all times* t. To be a *general* solution, the solution most also contain a set of *adjustable parameters* (unspecified constants which may take any value) equal in number to the sum of the orders of the highest derivatives appearing for each degree of freedom in the equation of motion.

In the example of the SHO, an example of a solution is

$$x(t) = x_{eq} + B\cos(\omega_0 t). \tag{6}$$

To verify this as a solution, we substitute it into the equation of motion (5). To do this, we first note that $x(t)-x_{eq} = A\cos(\omega_0 t)$ and $d^2x(t)/dt^2 = (d/dt)(dx(t)/dt) = (d/dt)(-\omega_0 A\sin(\omega_0 t)) = (-\omega_0 A)(\omega_0\cos(\omega_0 t)) = -\omega_0^2 A\cos(\omega_0 t)$. Thus, we have for (5),

$$-kB\cos(\omega_0 t) = -m\omega_0^2 B\cos(\omega_0 t).$$

The above equation will hold for all times t provided that $-k = -m\omega_0^2$, or equivalently,

$$\omega_0 = \sqrt{(}\frac{k}{m}). \tag{7}$$

Thus (6) qualifies as a solution, but only if ω_0 takes the value given in (7).

Note that because the equation is satisfied for only one value of ω_0, ω_0 does not count as an adjustable parameter. On the other hand, we do have a solution for all values of B. Thus the solution (6) has one adjustable parameter. To be a *general solution*, however, we here require two adjustable parameters because the only degree of freedom is x and the second derivative of x appears in (5).

A solution to (5) which does have two adjustable parameters, and therefore is a general solution, is

$$x(t) = x_{eq} + B\cos(\omega_0 t) + C\sin(\omega_0 t), \tag{8}$$

where (7) defines ω_0 and B and C are adjustable parameters. We leave the verification of this as a solution as a recommended exercise for the student.

4.4 Find a particular solution

The power of the general solution is that it gives us the ability to quickly find the solution $x(t)$ and therefore predict the future behavior of the system under any particular set of conditions. In order to solve for the adjustable parameters there typically must be one *condition* for each degree of freedom. For a mechanical system, often these are given as the *initial conditions*, the initial position *and velocity* of each particle in the system. The procedure for finding the particular solution is (1) write each condition in terms of the general solution, (2) solve the resulting set of equations for the adjustable parameters, and (3) write down the general solution while substituting the particular values found for the adjustable parameters. As an example, consider the SHO under the boundary conditions of initial position x_0 and initial velocity v_0 at time $t = 0$.

Step 1 — In terms of the general solution (8), the position at $t = 0$ is

$$\begin{aligned} x_0 \equiv x(t=0) &= x_{eq} + B\cos(\omega_0 \cdot 0) + C\sin(\omega_0 \cdot 0) \\ &= x_{eq} + B\cos(0) + C\sin(0) \\ &= x_{eq} + B(1) + C(0) \\ &= x_{eq} + B. \end{aligned} \tag{9}$$

Similarly, we may find the velocity at $t = 0$ from the definition $v(t) = dx(t)/dt = -\omega_0 B \sin(\omega_0 t) + \omega_0 C \cos(\omega_0 t)$,

$$\begin{aligned} v_0 \equiv v(t=0) &= -\omega_0 B \sin(\omega_0 0) + \omega_0 C \cos(\omega_0 0) \\ &= -\omega_0 B \sin(0) + \omega_0 C \cos(0) \\ &= -\omega_0 B(0) + \omega_0 C(1) \\ &= \omega_0 C. \end{aligned} \quad (10)$$

Step 2 — Next, solving (9-10) for the adjustable parameters, we have

$$\begin{aligned} B &= x_0 - x_{eq} \\ C &= v_0/\omega_0 \end{aligned} \quad (11)$$

Step 3 — Finally, we substitute these results into the general solution (8) to find the particular solution,

$$x(t) = x_{eq} + (x_0 - x_{eq}) \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \quad (12)$$

5 Complex representation

To introduce the use of the complex representation as a powerful method for the analysis of problems with cyclic behavior, we shall solve a problem first using the traditional approach and then using complex analysis. For the problem, let us ask for the amplitude A and initial phase ϕ_0 of a simple harmonic oscillator with initial conditions x_0 and v_0. Note that the traditional analysis here will get somewhat involved. What we wish to underscore here is that nothing mysterious underlies the complex approach but that it does make the analysis significantly simpler once you become used to it.

5.1 Traditional solution

From the definitions in Section 3.2, we can compute the amplitude as $A = x_{\text{max}} - x_{eq} \equiv x(t_{\text{max}}) - x_{eq}$ and the initial phase as $\phi_0 = \omega_0 t_{\text{el}} = \omega_0 (0 - t_{\text{max}}) = -\omega_0 t_{\text{max}}$, where $x(t)$ is given by our particular solution (12). In both cases, we need to determine the time t_{max} at which the maximum of $x(t)$ occurs. This we can do by setting the derivative to zero,

$$\begin{aligned} 0 &= \frac{dx(t)}{dt} \\ &= -\omega_0 (x_0 - x_{eq}) \sin(\omega_0 t) + \omega_0 (v_0/\omega_0) \cos(\omega t) \\ &= -\omega_0 (x_0 - x_{eq}) \sin(\omega_0 t) + v_0 \cos(\omega t), \end{aligned}$$

and solving for the time t,

$$\begin{aligned} \omega_0 (x_0 - x_{eq}) \sin(\omega_0 t) &= v_0 \cos(\omega t), \\ \frac{\sin(\omega_0 t)}{\cos(\omega_0 t)} &= \frac{v_0/\omega_0}{x_0 - x_{eq}} \\ \tan(\omega_0 t) &= \frac{v_0/\omega_0}{x_0 - x_{eq}} \\ &\Rightarrow \\ \omega_0 t_{\text{max}} &= \arctan \frac{v_0/\omega_0}{x_0 - x_{eq}}. \end{aligned} \quad (13)$$

According to our definition, the initial phase is therefore

$$\phi_0 = -\omega_0 t_{\text{max}} = -\arctan \frac{v_0/\omega_0}{x_0 - x_{eq}}, \quad (14)$$

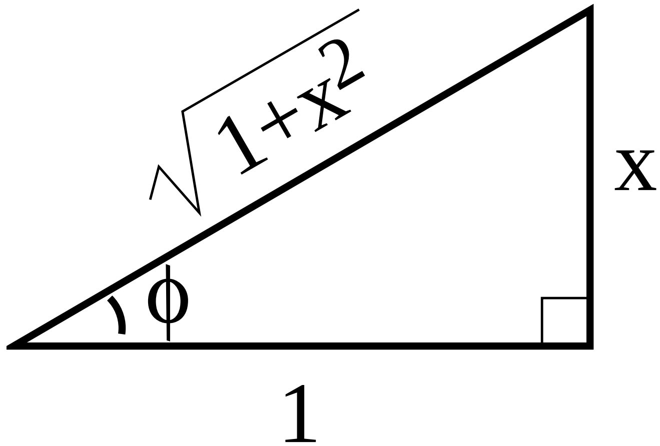

Figure 3: Right triangle for deriving trigonometric identities

and the amplitude is

$$\begin{aligned} A &= x(t_{\max}) - x_{eq} \\ &= (x_0 - x_{eq})\cos(\omega_0 t) + (v/\omega_0)\sin(\omega_0 t) \\ &= (x_0 - x_{eq})\cos(\arctan\frac{v_0/\omega_0}{x_0 - x_{eq}}) + (v/\omega_0)\sin(\arctan\frac{v_0/\omega_0}{x_0 - x_{eq}}), \end{aligned} \tag{15}$$

where, in the last step, we have substituted our result (13).

Further simplification of (15) is somewhat tricky because we must have knowledge of trig identities for $\cos(\arctan x))$ and $\sin(\arctan x))$. The standard trick to evaluate these is to note that the tangent of the angle ϕ in Figure 3 is x and therefore that

$$\begin{aligned} \cos(\arctan x)) &= \frac{1}{\sqrt{1+x^2}} \\ \sin(\arctan x)) &= \frac{x}{\sqrt{1+x^2}}. \end{aligned} \tag{16}$$

Thus,

$$\begin{aligned} (x_0 - x_{eq})\cos(\arctan\frac{v_0/\omega_0}{x_0 - x_{eq}}) &= (x_0 - x_{eq})\frac{1}{\sqrt{1+\left(\frac{v_0/\omega_0}{x_0 - x_{eq}}\right)^2}} \\ &= (x_0 - x_{eq})\frac{(x_0 - x_{eq})}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}}, \\ &= \frac{(x_0 - x_{eq})^2}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} (v/\omega_0)\sin(\arctan\frac{v_0/\omega_0}{x_0 - x_{eq}}) &= (v/\omega_0)\frac{\frac{(v/\omega_0)}{x_0 - x_{eq}}}{\sqrt{1+\left(\frac{v_0/\omega_0}{x_0 - x_{eq}}\right)^2}} \\ &= (v/\omega_0)\frac{(v/\omega_0)}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}} \end{aligned}$$

$$= \frac{(v/\omega_0)^2}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}}. \qquad (18)$$

Substituting (17-18) into (15), we have the final result for A.

$$\begin{aligned} A &= \frac{(x_0 - x_{eq})^2}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}} + \frac{(v/\omega_0)^2}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}} \\ &= \frac{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}{\sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2}} \\ &= \sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2} \qquad (19) \end{aligned}$$

Eqs.(14) and (19) answer the question of finding the amplitude and initial phase of an oscillator with initial position x_0 and velocity v_0. Now that we have expressions for the amplitude and phase, some additional trigonometric identities and transformations are worth noting. Substituting (14) and (19) into (17) and canceling one factor of $(x_0 - x_{eq})$ from both sides, we find

$$x_0 - x_{eq} = A\cos(-\phi_0) = A\cos(\phi_0). \qquad (20)$$

Working similarly with (14), (19) and (18),

$$v_0/\omega_0 = A\sin(-\phi_0) = -A\sin(\phi_0). \qquad (21)$$

Substituting these last two results into the general solution, we find an alternate form for the general solution in which A and ϕ_0 are the two adjustable parameters,

$$\begin{aligned} x(t) &= x_{eq} + (x_0 - x_{eq})\cos(\omega_0 t) + (v/\omega_0)\sin(\omega_0 t) \\ &= x_{eq} + A\cos(\phi_0)\cos(\omega_0 t) - A\sin(\phi_0)\sin(\omega_0 t) \\ &= x_{eq} + A\left(\cos(\phi_0)\cos(\omega_0 t) - \sin(\phi_0)\sin(\omega_0 t)\right) \\ &= x_{eq} + A\cos(\omega_0 t + \phi_0). \qquad (22) \end{aligned}$$

5.2 Complex analysis

The simple form of the final results in Section 5.1, Eqs. (14,19,22), strongly suggest that there should be a simpler analysis leading to the same conclusions. That simpler analysis is the subject of this section.

For this analysis, we find yet another form of the general solution. In particular, through the use of Euler's famous formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \qquad (23)$$

we may write the general solution (22) as

$$x(t) = x_{eq} + \Re\left(Ae^{i(\omega_0 t + \phi_0)}\right), \qquad (24)$$

where $\Re(\underline{z})$ denotes the "real part" of the complex number $\underline{z}$. Note that throughout these notes we shall underline variables to emphasize that they represent complex numbers.

The great utility of the complex representation is that it allows us to encode complicated trigonometric manipulations, such as those of the previous section, into familiar algebraic manipulations. For instance, we can split the exponent in (24) just as we would a real exponent,

$$\begin{aligned} x(t) &= x_{eq} + \Re\left(Ae^{i(\omega_0 t + \phi_0)}\right) \\ x(t) &= x_{eq} + \Re\left(Ae^{i\omega_0 t + i\phi_0}\right) \\ &= x_{eq} + \Re\left(Ae^{i\phi_0}e^{i\omega_0 t}\right) \\ &= x_{eq} + \Re\left(\underline{A}e^{i\omega_0 t}\right), \qquad (25) \end{aligned}$$

where in the last step we have combined the two real parameters of amplitude A and phase ϕ_0 into a single, handy *complex amplitude*

$$\underline{\mathrm{A}} = Ae^{i\phi_0}. \tag{26}$$

Now, to find the amplitude and phase, we repeat the general prescription for finding a particular solution from Section 4.4. We begin by expressing the boundary conditions in terms of the general solution,

$$\begin{aligned} x_0 = x(t=0) &= x_{eq} + \Re\left(\underline{\mathrm{A}}e^{i\omega_0\cdot 0}\right) \\ &= x_{eq} + \Re\left(\underline{\mathrm{A}}\cdot e^{0}\right) \\ &= x_{eq} + \Re\left(\underline{\mathrm{A}}\cdot 1\right) \\ &= x_{eq} + \Re\left(\underline{\mathrm{A}}\right), \end{aligned} \tag{27}$$

and

$$\begin{aligned} v_0 = \left(\left.\frac{dx}{dt}\right|_{t=0}\right. &= \left.\left(\Re\left(\underline{\mathrm{A}}i\omega_0 e^{i\omega_0 t}\right)\right|_{t=0}\right. \\ &= \Re\left(\underline{\mathrm{A}}i\omega_0 e^{i\omega_0\cdot 0}\right) \\ &= \Re\left(\underline{\mathrm{A}}i\omega_0\right) \\ &= \omega_0\Re\left(\underline{\mathrm{A}}i\right). \end{aligned} \tag{28}$$

The next step is to solve these equations for the undetermined parameters. In this case, because we have combined both the amplitude and phase into a single *complex* parameter $\underline{\mathrm{A}}$, there is only this one undetermined parameter. Now, Eq. (27) immediately gives us the real part of $\underline{\mathrm{A}}$,

$$\Re\left(\underline{\mathrm{A}}\right) = x_0 - x_{eq}. \tag{29}$$

Finding the imaginary part is just a little more subtle. Here, and throughout this course, when we need to talk about the real and imaginary parts of a complex number, we shall use the subscripts r and i respectively. Thus, we can write $\underline{\mathrm{A}} = A_r + A_i i$, so that $\Re\left(\underline{\mathrm{A}}\right) = A_r$ and $\Im\left(\underline{\mathrm{A}}\right) = A_i$. Using this notation, Eq. (28) becomes

$$\begin{aligned} v_0 &= \omega_0\Re\left(\left(\left(A_r + A_i i\right) i\right)\right. \\ &= \omega_0\Re\left(A_r i - A_i\right) \\ &= -\omega_0 A_i. \end{aligned}$$

Thus, we also have the imaginary part of $\underline{\mathrm{A}}$,

$$\Im\left(\underline{\mathrm{A}}\right) = -v_0/\omega_0. \tag{30}$$

Putting both (29) and (30) together, we have the complex amplitude,

$$\underline{\mathrm{A}} = (x_0 - x_{eq}) - \frac{v_0}{\omega_0}i. \tag{31}$$

Finally, it is our job to find the real amplitude A and initial phase ϕ_0 given the above value for the complex amplitude $\underline{\mathrm{A}}$.

To get the amplitude A and phase ϕ_0 for any complex number $\underline{\mathrm{A}}$, we use Euler's formula (23),

$$\begin{aligned} \underline{\mathrm{A}} &= Ae^{i\phi_0} \\ &= A\left(\cos\phi_0 + i\sin\phi_0\right) \\ &\Rightarrow \\ A_r &= A\cos\phi_0 \\ A_i &= A\sin\phi_0. \end{aligned} \tag{32}$$

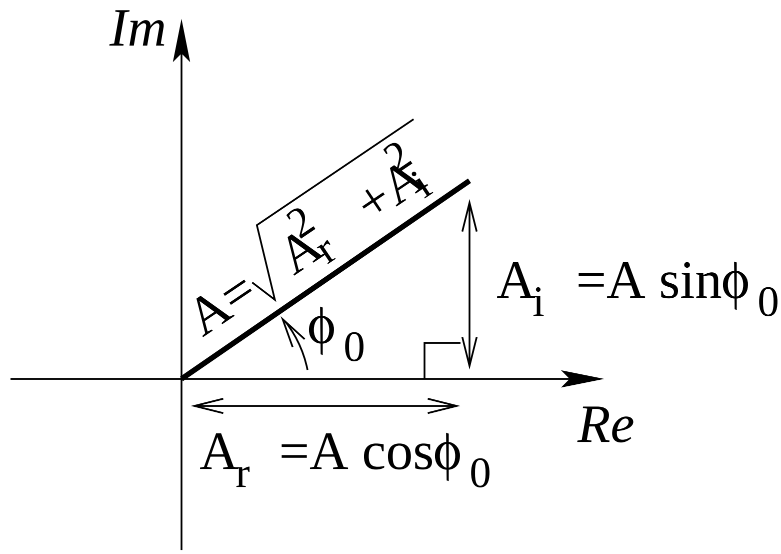

Figure 4: Argand diagram: connecting real and imaginary parts to amplitude and phase of a complex number

We see that A_r is the adjacent side to angle ϕ_0 in a right triangle of hypotenuse A while A_i is the opposite side of the same triangle. (See Figure 4.) Thus, we may always determine the amplitude and phase as

$$\begin{aligned} A \;\; (\equiv |\underline{\mathrm{A}}|) &= \sqrt{{A_r}^2 + {A_i}^2} \\ \phi_0 \;\; \left(\equiv \phi_{\underline{\mathrm{A}}}\right) &= \arctan \frac{A_i}{A_r}, \end{aligned} \tag{33}$$

where we have introduced the notations $|\underline{\mathrm{A}}|$ and $\phi_{\underline{\mathrm{A}}}$ for the amplitude and phase of the complex number $\underline{\mathrm{A}}$, respectively.

Finally, using our result for $\underline{\mathrm{A}}$ (31) and the conversions (33), we quickly find the real amplitude and initial phase of an oscillator with initial position x_0 and initial velocity v_0,

$$\begin{aligned} A &= \sqrt{(x_0 - x_{eq})^2 + (v_0/\omega_0)^2} \\ \phi_0 &= -\arctan \frac{v_0/\omega_0}{x_0 - x_{eq}} \end{aligned}$$

These results are exactly what we found previously (14,19), but now derived much more easily!!!

6 Driven Oscillators

6.1 Motivation

In this section, we would like to apply what we have learned in the previous sections to address the question of what happens when we take an oscillator and push, or *drive*, it with a periodic force. This is somewhat akin to pushing a child on a swing, where the swing by itself is an oscillatory system with a certain *natural* frequency ω_0, and we can drive the system by pushing with whatever frequency ω we desire. An interesting questions about this system is what driving frequency will result in the largest response? As you may suspect, the answer is to drive at a frequency which nearly matches the natural frequency of the system, $\omega \approx \omega_0$. But, as we shall see, the answer isn't quite that simple.

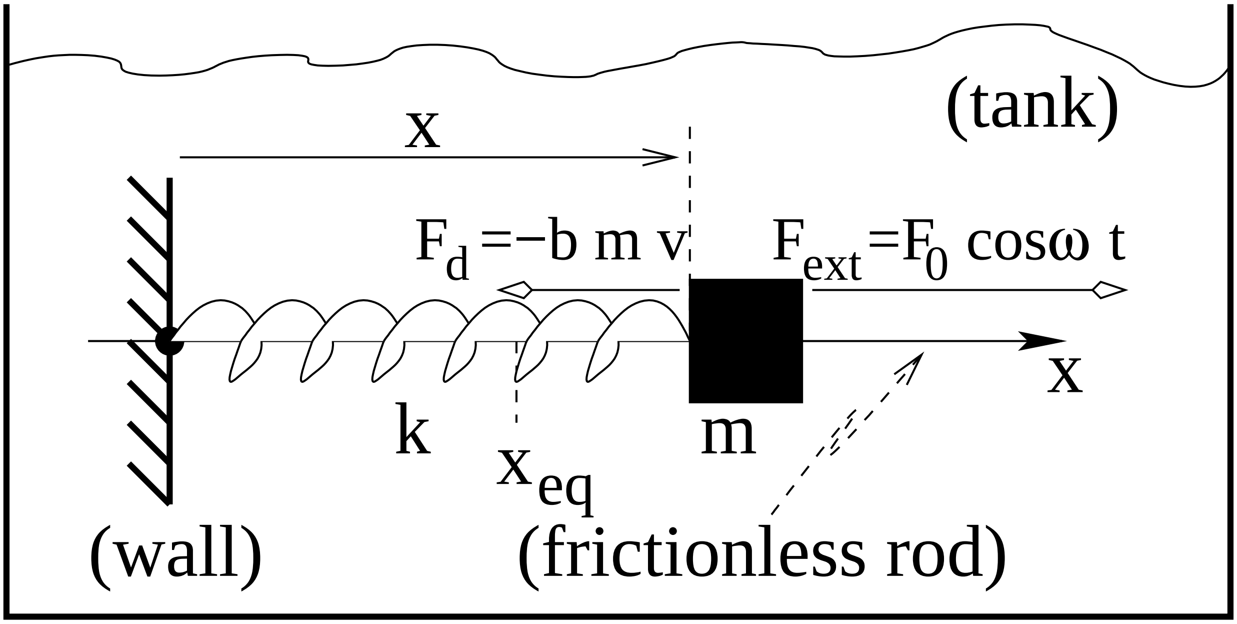

Figure 5: Damped, driven simple harmonic oscillator: spring-mass system with an external driving force and damping

6.2 Physical Realization

Figure 5 shows our physical realization of a damped, driven harmonic oscillator. Note that the system is very similar to the simple harmonic oscillator in Figure 1. The only new physical forces at work are the driving force $F_{\text{ext}} = F_0 \cos \omega t$ with magnitude F_0, initial phase $\phi_F = 0$ and frequency ω, and a viscous drag force $F_d = -bmv$ which is in direct proportion but oppositely oriented to the velocity, always tending to slow or *damp* the motion.

We consider damping in this problem because (a) all realistic physical systems involve some sort of damping, (b) while damping may be ignored in many applications, it becomes particularly important when we drive at the natural frequency and thereby generate large, rapid motions. Note that we write the damping coefficient as bm purely as a mathematical convenience. If you prefer, you can think of the drag force as being $F_d = -\alpha v$, and then the value of b is simply $b = \alpha/m$.

Finally, we use ω to indicate the driving frequency. This is something entirely different from the natural frequency of the oscillator $\omega_0 \equiv \sqrt{k/m}$, which we define with the subscript "0".

6.3 Analysis

We now repeat the stages of our general analysis.

1. Degrees of Freedom (D of F): As with the simple harmonic oscillator, the only degree of freedom is x.

2. Equation of Motion (E of M): Starting from Newton's law and continuing until we have only x, derivatives of x and constants defined in the problem, we find

$$\begin{aligned} \sum F_x &= ma_x \\ -k(x - x_{eq}) + F_{\text{ext}} + F_d &= m\frac{d^2x}{dt^2} \end{aligned}$$

$$\begin{aligned}
-k(x - x_{eq}) + F_0 \cos\omega t - bmv &= m\frac{d^2x}{dt^2} \\
-k(x - x_{eq}) + F_0 \cos\omega t - bm\frac{dx}{dt} &= m\frac{d^2x}{dt^2}.
\end{aligned}$$

This now is a valid equation of motion. Simplifying by dividing through by m and rearranging a bit, we find

$$\begin{aligned}
-\frac{k}{m}(x - x_{eq}) + \frac{F_0}{m}\cos\omega t - b\frac{dx}{dt} &= \frac{d^2x}{dt^2} \\
\Rightarrow \quad \frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2(x - x_{eq}) &= \frac{F_0}{m}\cos\omega t, \qquad (34)
\end{aligned}$$

where we have defined the natural frequency of the system as $\omega_0 \equiv \sqrt{k/m}$ as above in (7).

3. Solution: Here, we seek only a typical solution to the equation of motion. A reasonable guess is that the mass will oscillate about the equilibrium point at the driving frequency with some amplitude A and initial phase ϕ_0. As we anticipate the complex representation making our mathematical task much easier, we also express this guess solution in complex form. Our guess solution is therefore

$$\begin{aligned}
x(t) &= x_{eq} + A\cos(\omega t + \phi_0) \qquad (35) \\
&= x_{eq} + \Re\left(Ae^{i(\omega t + \phi_0)}\right) \\
&= x_{eq} + \Re\left(\underline{\mathrm{A}}e^{i\omega t},\right) \qquad (36)
\end{aligned}$$

where, as above, we define $\underline{\mathrm{A}} \equiv Ae^{i\phi_0}$.

To verify this as a solution and to identify the appropriate value of $\underline{\mathrm{A}}$, we substitute (36) into the equation of motion. As usual, we first evaluate the useful terms before substituting:

$$\begin{aligned}
x - x_{eq} &= \Re\left(\underline{\mathrm{A}}e^{i\omega t}\right) \\
\frac{dx}{dt} &= 0 + \Re\left(\underline{\mathrm{A}}i\omega e^{i\omega t}\right) \\
&= \Re\left(\underline{\mathrm{A}}i\omega e^{i\omega t}\right) \\
\frac{d^2x}{dt^2} &= \Re\left((\underline{\mathrm{A}}i\omega)i\omega e^{i\omega t}\right) \\
&= \Re\left(-\omega^2\underline{\mathrm{A}}e^{i\omega t}\right) \\
\frac{F_0}{m}\cos\omega t &= \Re\left(\frac{F_0}{m}e^{i\omega t}\right),
\end{aligned}$$

where, in the last step, we use Euler's formula to get the driving force term into a form similar to all of the other terms. Substituting all of these terms into the equation of motion (34), and collecting terms we find

$$\begin{aligned}
\Re\left(-\omega^2\underline{\mathrm{A}}e^{i\omega t}\right) + b\Re\left(\underline{\mathrm{A}}i\omega e^{i\omega t}\right) + \omega_0^2\Re\left(\underline{\mathrm{A}}e^{i\omega t}\right) &= \Re\left(\frac{F_0}{m}e^{i\omega t}\right) \\
\Re\left(\left[\left\{-\omega^2 + ib\omega + \omega_0^2\right\}\underline{\mathrm{A}} - \frac{F_0}{m}\right]e^{i\omega t}\right) &= 0. \qquad (37)
\end{aligned}$$

To complete the argument, we note that the left hand side of (37) describes an oscillation of amplitude $|\left\{-\omega^2 + ib\omega + \omega_0^2\right\}\underline{\mathrm{A}} - \frac{F_0}{m}|$, whereas the right hand size is zero. This equation holds for all time t if and only if the amplitude of the oscillation on the left-hand size is zero. Thus, the equation of motion is satisfied by our solution, but only if

$$\left\{-\omega^2 + ib\omega + \omega_0^2\right\}\underline{\mathrm{A}} - \frac{F_0}{m} = 0,$$

or, solving for $\underline{\mathrm{A}}$,

$$\underline{\mathrm{A}} = \frac{\frac{F_0}{m}}{-\omega^2 + ib\omega + \omega_0^2} \tag{38}$$

4. Magnitude and Phase: From the result (41), we are now in a position to determine the amplitude and initial phase of the resulting motion. We know that these, respectively, are the magnitude and phase of the complex number $\underline{\mathrm{A}}$. To simplify our work, we note that (38) has the form of a quotient of two complex numbers. The magnitude and phase of such a quotient may be determined simply from the following argument,

$$\begin{aligned} \underline{\mathrm{A}} &= \frac{\underline{\mathrm{B}}}{\underline{\mathrm{C}}} \\ &= \frac{|\underline{\mathrm{B}}| e^{i\phi_{\underline{\mathrm{B}}}}}{|\underline{\mathrm{C}}| e^{i\phi_{\underline{\mathrm{C}}}}} \\ &= \left(\frac{|B|}{|C|}\right) e^{i(\phi_{\underline{\mathrm{B}}} - \phi_{\underline{\mathrm{C}}})}, \end{aligned}$$

where we have used the fact that complex numbers $\underline{\mathrm{B}}$ and $\underline{\mathrm{C}}$ always may be decomposed into amplitudes and phases, $\underline{\mathrm{B}} \equiv |\underline{\mathrm{B}}| \exp(i\phi_{\underline{\mathrm{B}}})$ and $\underline{\mathrm{C}} \equiv |\underline{\mathrm{C}}| \exp(i\phi_{\underline{\mathrm{C}}})$. From the final line of (39), we learn two simple rules: (1) the magnitude of a quotient is the quotient of the magnitudes, and (2) the phase of a quotient is the *difference* of the phases. Mathematically,

$$\left|\frac{\underline{\mathrm{B}}}{\underline{\mathrm{C}}}\right| = \frac{|\underline{\mathrm{B}}|}{|\underline{\mathrm{C}}|} \tag{39}$$

$$\phi_{\underline{\mathrm{B}}/\underline{\mathrm{C}}} = \phi_{\underline{\mathrm{B}}} - \phi_{\underline{\mathrm{C}}} \tag{40}$$

Thus, in our particular case,

$$\begin{aligned} A &= \frac{\left|\frac{F_0}{m}\right|}{|-\omega^2 + ib\omega + \omega_0^2|} \\ &= \frac{\frac{F_0}{m}}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b\omega)^2}}, \end{aligned} \tag{41}$$

and,

$$\begin{aligned} \phi_0 &= 0 - \arctan \frac{b\omega}{\omega_0^2 - \omega^2} \\ &= -\arctan \frac{b\omega}{\omega_0^2 - \omega^2}. \end{aligned} \tag{42}$$

Figures 6 and 7 plot these results for the amplitude and initial phase, respectively.

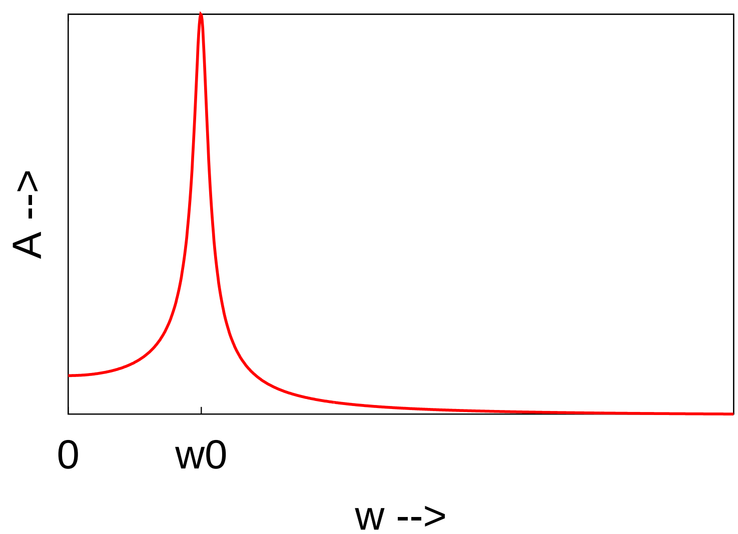

Figure 6: Amplitude (A) of driven, damped harmonic oscillator as a function of drive angular frequency (w)

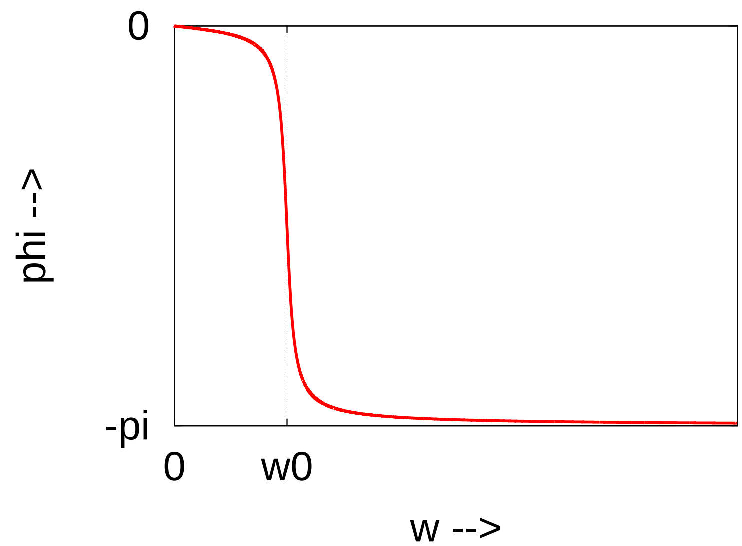

Figure 7: Initial phase (phi) of driven, damped harmonic oscillator as a function of drive frequency (w)