Class Notes VI: Transport of momentum and energy

November 30, 2001

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Department of Physics

Physics 214 November 30, 2001

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1 Introduction

These notes describe how waves transport momentum and energy without the net movement of particles from one end of the system to another. From basic principles, we expect waves to conserve momentum and energy (as well as angular momentum). What is not so clear upfront is that a picture of flow of these quantities from one neighboring point to the next accounts for this conservation. The fact that energy and momentum indeed flow from one point to the next allows us to understand how waves propagate their influence across space.

2 Continuity Equation

The general *continuity equation* accounts for the conservation of any quantity Q which may be flowing in a system in terms of the density of that quantity at all points $q(x)$ and the rate of flow of the quantity $F(x)$ from left-to-right from one point to the next.

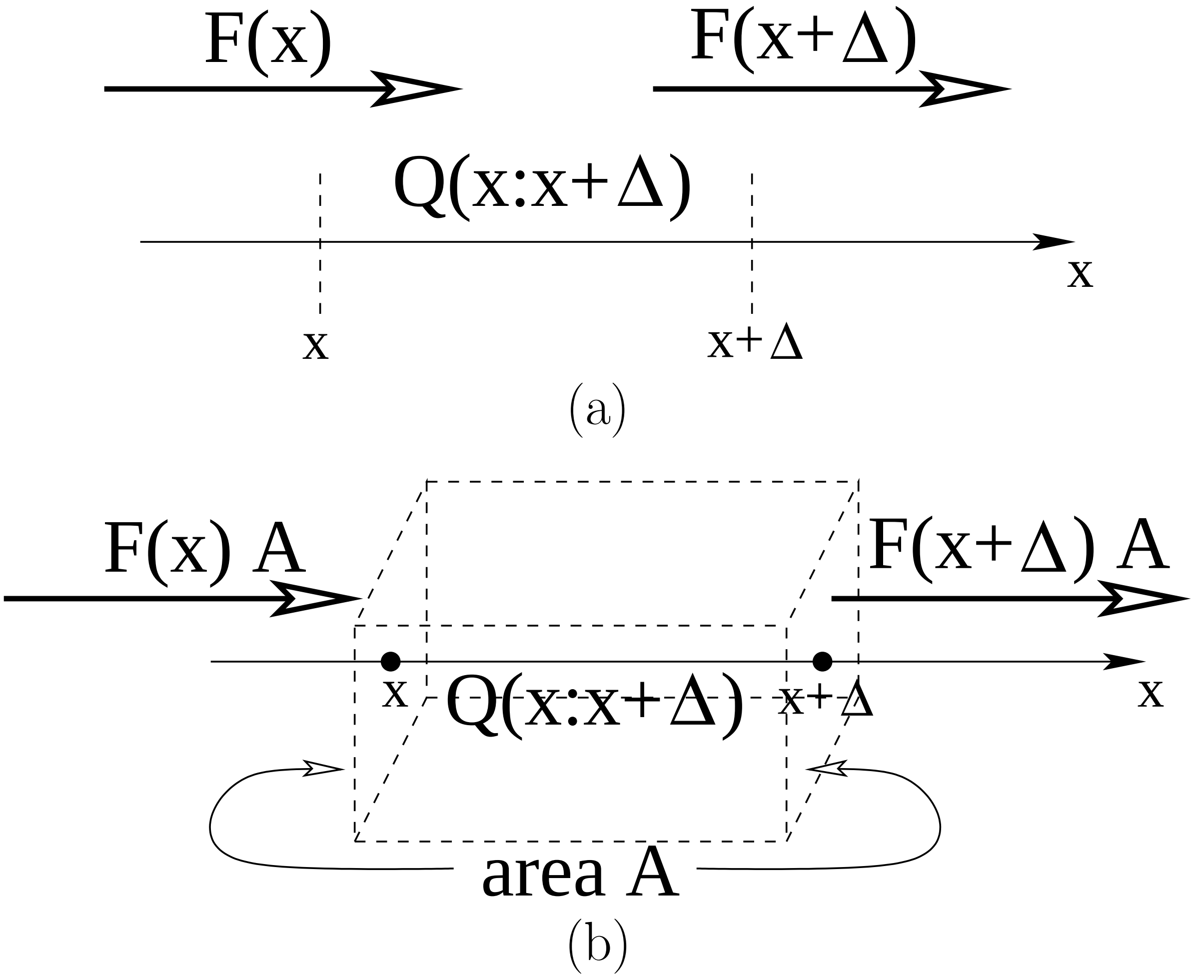

Figure 1: Conservation of quantity Q stored between points x and $x + \Delta$ in terms of in-flow and out-flux F(x): (a) in $d = 1$ dimension, and (b) in $d = 3$ dimensions.

If we indeed Q is conserved while flowing from point to neighboring point, then the rate of change in the amount of Q between points x and $x+\Delta$ comes only from flow at points x and $x+\Delta$, as Figure 1 illustrates.

The mathematical definitions of the relevant quantities depend upon the dimensionality of the system. If we describe a $d=1$ dimensional system (Figure 1a), then we define the density q as the linear density, the amount of quantity Q *per unit length*, and the flow $F(x)$ as the *rate of flow* of Q across the point x. By convention, we define flow from left to right as positive. Then, if $Q(x:x+\Delta)$ is the amount of quantity Q between points x and $x+\Delta$, we have a contribution flowing *in* at point x at a rate of $+F(x)$, and a contributing flowing *out* at point $x+\Delta$ at a rate $F(x+\Delta)$. (See Figure.) Thus,

$$\frac{\partial}{\partial t}Q(x:x+\Delta) = F(x) - F(x+\Delta), \tag{1}$$

expresses conservation of Q in between points x and $x+\Delta$.

A more useful expression of the same idea is one in differential form. To find this, we divide (1) through by Δ, rearrange the order of terms on the right-hand side, and take the limit $\Delta \to 0$. This gives

$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q(x:x+\Delta)}{\Delta} &= -\frac{F(x+\Delta)-F(x)}{\Delta} \\ &(\Delta \to 0) \\ \frac{\partial}{\partial t}q(x) &= -\frac{\partial}{\partial x}F(x).\end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}q + \frac{\partial}{\partial x}F = 0. \tag{2}$$

If, on the other hand, we have a $d=3$ dimensional system (Figure 1b), we then define the density q as the volume density, the amount of quantity Q *per unit volume*, and the flow F as the rate of flow *per unit area* of quantity Q. Following a similar analysis, we will now find that, under these definitions, the same mathematical expression applies also in $d=3$ dimensions. Note that we continue to assume plane-wave behavior so that the basic system quantities depend only upon x.

To carry out the analysis, we now define $Q(x:x+\Delta)$ as the amount of the quantity in the *box* of cross-sectional area A extending from point x to point $x+\Delta$. The net rate of flow *into* the box across the face at point x is now $F(x)A$ and the flow *out of* the box across the face at $x+\Delta$ is $F(x+\Delta)A$. Thus,

$$\begin{aligned}\frac{\partial}{\partial t}Q(x:x+\Delta) &= F(x)A - F(x+\Delta)A \\ \frac{\partial}{\partial t}\frac{Q(x:x+\Delta)}{\Delta\cdot A} &= -\frac{F(x+\Delta)-F(x)}{\Delta} \\ \frac{\partial}{\partial t}\frac{Q(x:x+\Delta)}{V} &= -\frac{F(x+\Delta)-F(x)}{\Delta} \\ &(\Delta \to 0) \\ \frac{\partial}{\partial t}q(x) &= -\frac{\partial}{\partial x}F(x)\end{aligned}$$

Hence, we again find the same equation, but with the quantities now defined as appropriate for a $d=3$ dimensional system,

$$\frac{\partial}{\partial t}q + \frac{\partial}{\partial x}F = 0. \tag{3}$$

The identical equations (2,3) are known as *The Continuity Equation* for quantity Q. They apply for any quantity which is conserved and which moves by flowing from one point to the neighboring point. We now consider whether momentum and energy can be described in this way.

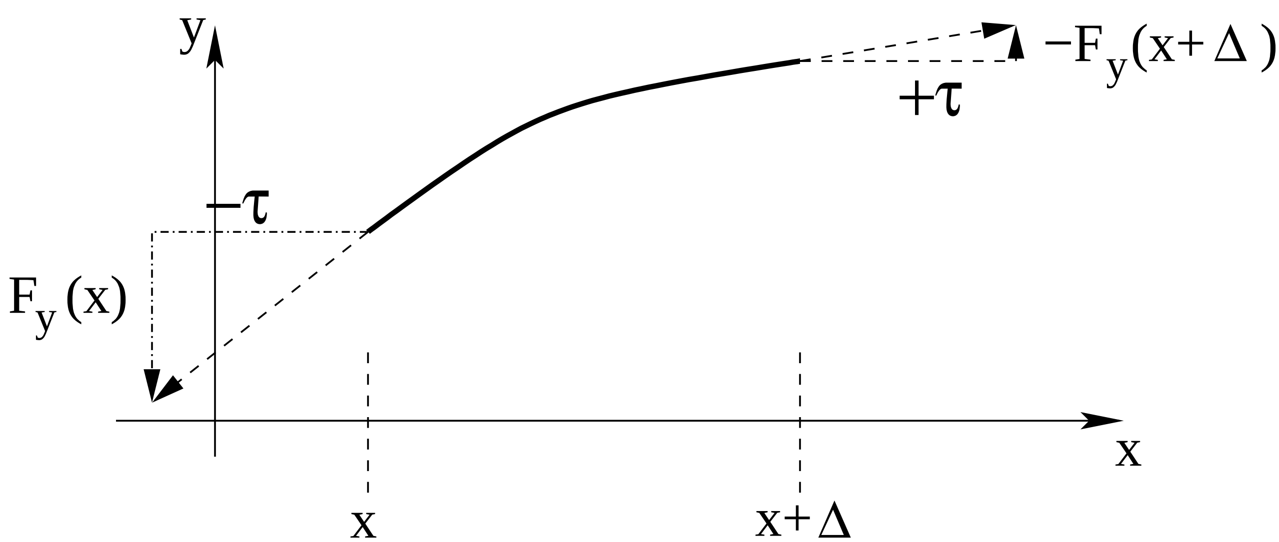

Figure 2: Momentum and energy flow via forces on a string

3 Conservation of Momentum

3.1 Strings

We begin by considering the case of the string. All motion is then along the y direction; hence, we need consider only momentum in the y-direction, P_y. The string is a $d = 1$ dimensional system and we therefore consider the linear momentum density p_y, the momentum per unit length. Momentum being mass times velocity, the momentum per unit length is the mass per unit length μ times the velocity; hence,

$$p_y = \mu \frac{\partial y}{\partial t}, \tag{4}$$

plays the role of q in (2) when considering conservation of momentum[1].

Next, to understand the flow of momentum, we note that Newton's law for finite systems states

$$\sum \vec{F}^{(ext)} = m_{sys} \vec{a}_{c\,of\,m} = \frac{d}{dt}\left(m_{sys}\vec{v}_{c\,of\,m}\right) = \frac{d}{dt}\vec{P}^{(sys)}.$$

Thus, the rate of change of the total momentum of any system (such as the chunk between points x and $x+\Delta$) is the sum of *external* forces acting on it. As we have always ignored gravity (and the other long-range forces) in our analyses, the only relevant remaining forces are the contact forces at points x and $x + \Delta$.

For the string (Figure 2), the contact force at point x comes from the tension in the segment of the string to the left, $-\tau\hat{x} - \tau(\partial y(x)/\partial x)\hat{y}$, and the force at point $x + \Delta$ comes from the tension in the segment to the right $+\tau\hat{x} + \tau(\partial y(x + \Delta)/\partial x)$. Consistent with what we already know about lack of x motion in strings, the x components of the tensions cancel, leaving us to consider motion in the y direction only. If we now define the quantity

$$F_y(x) \equiv -\tau(\partial y(x)/\partial x) \tag{5}$$

[1] As a practical aside, note that if we ever require the total y-momentum stored between points a and b along the string, we take the density at each chunk p_y, multiply by the length of the chunk dx and sum over all chunks,

$$P_y = \int_a^b p_y\, dx = \int_a^b \mu \frac{\partial y}{\partial t}\, dx.$$

for any point x, we may then write

$$\frac{d}{dt}P_y^{(sys)} = \sum F_y^{(ext)} = F_y(x) - F_y(x+\Delta). \tag{6}$$

Comparing (6) with (1), we see that the quantity describing the left-right flow $F(x)$ across point x in (1) is nothing other than the y component of the tension force $F_y(x)$, with the sign as in (5). Thus, we may interpret force as describing the rate of flow of momentum from one part of a system to another.

We are now ready to consider the continuity equation for momentum along a string. The quantity p_y from Eq. (4) plays the role of q, the density of the conserved quantity, and the quantity $F_y(x)$ from Eq. (5) plays the role of $F(x)$, the rate of flow of the conserved quantity. Thus, *if indeed momentum is conserved for a string*, we expect the following equation to hold,

$$\frac{\partial}{\partial t}p_y + \frac{\partial}{\partial x}F_y(x) =^{(?)} 0. \tag{7}$$

To verify whether this equation holds, we substitute our definitions (4,5) of the density and flow into (7) to find

$$\frac{\partial}{\partial t}\left(\mu\frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial x}\left(-\tau\frac{\partial y}{\partial x}\right) = 0, \tag{8}$$

which we know to be true because we recognize it as the wave equation for the string!!!

3.2 Sound

Continuity in the case of sound works very similarly. As sound is longitudinally polarized, the direction of motion is now along the x direction, and we thus consider this component of the momentum. Following the interpretation of quantities in (3), $q(x)$ should now represent the *volume* density of momentum, and $F(x)$ should represent the rate of flow of momentum *per cross sectional area* from left to right across point x.

The volume density of momentum is the mass of each chunk times its velocity divided by its volume, namely the mass per unit volume ρ_0 times the velocity. Thus,

$$p_x = \rho_0\frac{\partial s}{\partial t}. \tag{9}$$

From the discussion of the string, we know that *force* is the measure of the rate of momentum flow. Because sound occurs in three dimensions, $F(x)$ should measure flow *per unit area*. Thus, $F(x)$ should be force per unit area, the *pressure*

$$P(x) = P_0 - B\frac{\partial s}{\partial x}. \tag{10}$$

To verify the correct sign, note that the pressure at x pushes in the positive direction on the chunk to its right, and thus $F(x) = +P(x)$.

Substituting these results for $q(x)$ and $F(x)$ into (3), the three-dimensional continuity equation for momentum for sound is

$$\begin{aligned} \frac{\partial}{\partial t}\left(\rho_0\frac{\partial s}{\partial t}\right) + \frac{\partial}{\partial x}\left(P_0 - B\frac{\partial s}{\partial x}\right) &= 0 \\ \frac{\partial}{\partial t}\left(\rho_0\frac{\partial s}{\partial t}\right) + \frac{\partial}{\partial x}\left(-B\frac{\partial s}{\partial x}\right) &= 0, \end{aligned} \tag{11}$$

which we know to be true because we again recognize it as the wave equation!!!

4 Conservation of Energy

4.1 Strings

To express conservation of energy, we need both the energy density, which we shall call $e(x)$, and the rate of flow of energy, which we shall call the power $\mathcal{P}(x)$.

The left-right rate of flow of energy across point x is the rate at which the string to the left of point x does work on the string on the other side of x. (See Figure 2.) The work is the force times displacement, and so the rate of flow of energy is force times displacement per unit time, or simply the familiar power formula of force times velocity. The velocity is $v_y = (\partial y/\partial t)$, and the force exerted by the string to the left of x on the string to the right of x is, again, $F_y(x) = -\tau(\partial y/\partial x)$. Thus, the left-right rate of flow of energy is

$$\mathcal{P}(x) = -\tau \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}. \tag{12}$$

For $e(x)$, we note that energy generally consists of two forms, kinetic and potential, whose densities we shall write as $ke(x)$ and $pe(x)$, respectively, so that

$$e(x) = ke(x) + pe(x). \tag{13}$$

The kinetic energy density is the kinetic energy of a chunk divided by its length. The usual formula for kinetic energy is $KE = \frac{1}{2}mv^2$. If we divide this by the length of a chunk, the factor of m divided by length becomes the linear mass density μ. Using the fact that each chunk moves only along the y direction and the fact that $v_y = \partial y/\partial t$, we find

$$ke(x) = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2. \tag{14}$$

The proper form for the potential energy density is a tricky question. There is no obvious answer, and so we must return to the fundamental definition of potential energy as "stored" energy.

On the string, we must be careful in accounting for the energy because energy not only can be in the form of kinetic or potential energy but also can flow. Potential energy thus must be whatever is left over once we account for energy flow and kinetic energy. We now know the continuity equation as the proper way to do this accounting, and so we must find whether there is a form for $pe(x)$ that will make the following equation true,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}e(x) + \frac{\partial}{\partial x}\mathcal{P}(x) \\ &= \frac{\partial}{\partial t}\left(ke(x) + pe(x)\right) + \frac{\partial}{\partial x}\mathcal{P}(x) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + pe(x)\right) + \frac{\partial}{\partial x}\left(-\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right). \end{aligned} \tag{15}$$

Note that, other than our belief that the string should conserve energy, there is no guarantee that we will be able to find a formula for $pe(x)$ so that this equation will work. Thus, if we do find such a formula, we simultaneously accomplish two things: (a) we establish that the string does indeed conserve energy, and (b) we find the proper formula for the potential energy of the string.

To do this, we begin by taking the derivatives in (15),

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + pe(x)\right) + \frac{\partial}{\partial x}\left(-\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right) \\ &= \frac{1}{2}\mu\cdot 2\left(\frac{\partial y}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial t}pe(x) - \tau\left(\frac{\partial}{\partial x}\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial t} - \tau\frac{\partial y}{\partial x}\left(\frac{\partial}{\partial x}\frac{\partial y}{\partial t}\right), \end{aligned}$$

where, for the leftmost term on the first line, we used the chain rule to write the derivative of a square as twice the quantity being squared times the derivative of the quantity, and, for the rightmost term on the first line, we used the product rule to write the derivative of a product as the derivative of the first factor times the second factor plus the first factor time the derivative of the second factor. Simplifying and rearranging terms, we find

$$0 = \mu\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + \frac{\partial}{\partial t}pe(x) - \tau\frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t} - \tau\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x\partial t}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t} pe(x) + \frac{\partial y}{\partial t}\left(\mu \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2}\right) - \tau \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t} \\
&= \frac{\partial}{\partial t} pe(x) - \tau \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t},
\end{aligned}
$$

where the two terms in the large parentheses canceled as a result of the wave equation! This leaves only,

$$
\frac{\partial}{\partial t} pe(x) = \tau \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2\right),
$$

where the final, key, step on the right-hand side comes again from the chain rule for differentiating the square of a quantity. We now integrate both sides of this equation with respect to time to find

$$
pe(x) = \frac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2 + C(x),
$$

where $C(x)$ is an arbitrary integration constant with respect to time which may depend on position.

Several comments on the above result are in order. First, we have found a form for $pe(x)$, which means that we have proved that energy on the string is indeed conserved! Moreover, we find that the form of the potential energy density is related to the slope of the string. This is sensible because if the string is not flat, there is stored energy: upon release, the string will start to flatten, converting the stored potential energy into kinetic energy. It is also sensible that the potential energy goes like the square of the slope, as either a positive or negative slope should store the same positive amount of energy. We also understand the dependence on τ because the energy needed to create a given shape on the string should be in proportion to the overall tension. Finally, the arbitrary constant $C(x)$ represents the usual "arbitrary zero of potential energy" that one is always free to set. To keep things simple, we exploit our freedom and choose $C(x) = 0$, so that our final result for the potential energy is

$$
pe(x) = \frac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2. \tag{16}
$$

4.2 Sound and Electromagnetic waves

For sound and electromagnetic waves, one can carry out very similar derivations. The derivations, in fact, will be exactly the same with analogous quantities appearing in the corresponding places. Rather than repeat the same equations over and over, we choose here to show how to exploit the analogy. Following Eqs. (8) and (11) and the differential form of Ampere's law from our notes on electromagnetic waves, we have the following analogous equations for all three systems

System		inertia	velocity		elasticity	distortion	
String:	$0 = \frac{\partial}{\partial t}($	μ	$\frac{\partial y}{\partial t}$	$) - \frac{\partial}{\partial x}($	τ	$\frac{\partial y}{\partial x}$	$)$
Sound:	$0 = \frac{\partial}{\partial t}($	ρ_0	$\frac{\partial s}{\partial t}$	$) - \frac{\partial}{\partial x}($	B	$\frac{\partial s}{\partial x}$	$)$
E&M:	$0 = \frac{\partial}{\partial t}($	ϵ	E_z	$) - \frac{\partial}{\partial x}($	$\frac{1}{\mu}$	B_y	$)$

(17)

where analogous quantities are aligned in columns.

Next, we write our result for the conservation of energy for the string at the top of a new table, and produce valid equations for the other systems by substituting the analogous quantities from the columns of

(17) for each system,

$$
\begin{array}{r||c|c|c|c|c|c|c}
\text{System} & 0 = \frac{\partial}{\partial t}(& ke(x) & + & pe(x) &) + \frac{\partial}{\partial x}(& \mathcal{P}(x) &) \\
\text{String:} & 0 = \frac{\partial}{\partial t}(& \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 & + & \frac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2 &) + \frac{\partial}{\partial x}(& -\tau \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} &) \\
\text{Sound:} & 0 = \frac{\partial}{\partial t}(& \frac{1}{2}\rho_0 \left(\frac{\partial s}{\partial t}\right)^2 & + & \frac{1}{2}B \left(\frac{\partial s}{\partial x}\right)^2 &) + \frac{\partial}{\partial x}(& -B \frac{\partial s}{\partial x}\frac{\partial s}{\partial t} &) \\
\text{E\&M:} & 0 = \frac{\partial}{\partial t}(& \frac{1}{2}\epsilon E^2 & + & \frac{1}{2}\frac{1}{\mu}B^2 &) + \frac{\partial}{\partial x}(& \frac{1}{\mu}\vec{E} \times \vec{B} &)
\end{array}
\tag{18}
$$

All of the above equations are correct and represent the conservation of energy for the corresponding systems. Note that, for sound and E&M waves, which occur in three dimensions, the energy densities $ke(x)$ and $pe(x)$ are per unit *volume* (J/m^3), and the energy fluxes $\mathcal{P}(x)$ give power *per unit area* (Watt/m^2). This later quantity actually is the physical definition of *intensity*, the quantity which we studied in the previous unit in this course. The fact that the formulas for $\mathcal{P}(x)$ all contain a product of two wave variables explains why we took the "intensity" to be proportional to the *square* of the amplitude.

A few interpretive comments are important for the E&M results. In addition to the analogy, the table properly takes the vector nature of the electromagnetic field into account. The analogy tell us that we should put $(1/2)\epsilon E_z^2$ in the "$ke(x)$" column. However, this accounts only the energy from E_z. When all components are accounted, the final result involves the total square magnitude of the field, $E^2 \equiv E_x^2 + E_y^2 + E_z^2$ so that we put E^2 instead of just E_z^2. Similarly, for "$pe(x)$", the analogy directly tells us to put $(1/2)(1/\mu)B_y^2$, but to account for all the components we put $B^2 \equiv B_x^2 + B_y^2 + B_z^2$ in place of B_y^2. Finally, the analogy would have us put $-(1/\mu)EB$ for $\mathcal{P}(x)$. Accounting properly for all the signs involved in the equations for each of the components, we know that the result should point in the direction of travel of the wave, which we already learned to be along $\vec{E} \times \vec{B}$. Thus, the vector expression we give for $\mathcal{P}(x)$ is arranged to give both the correct magnitude and direction for the flow of energy.

As a final note, the result given for E&M is correct and represents conservation of energy, but the terms $\epsilon E^2/2$ and $(1/\mu)B^2/2$ do not represent "kinetic" and "potential"" energy in the usual sense. Rather, they give the energy density stored in the electric and magnetic fields, respectively. You should recall these energy density expressions as exactly the same expressions from your class in electromagnetism for the energy density stored, respectively, between the plates of a capacitor and inside of a solenoid!