Exercise Problems # 3

<u>**Note:**</u> Below is a list of some exercise problems that could be used as a supplement to your preparation for the exam. These problems have been given as homework or exam problems in previous years. Note that the list is <u>**not comprehensive**</u> and it is <u>**not meant to cover all the topics**</u> taught in the course so far. You are **primarily responsible for the material presented in Lectures (everything from diffraction on), Problem Sets 8—10, and Lab III.** You are expected to understand all the concepts in this material (including derivations) and to apply them creatively to different situations. See the **"Final Exam Study Tips"** announcement on the course web page.

1. Exercise Problem 1:

A single slit of width a is illuminated by an off-axis source. The light from the source has wavelength λ and hits the slit at an angle θ_0. Our goal in this problem is to find out how the single-slit diffraction pattern at a distance screen is changed by having the source off-axis (compared to the "regular" setup where $\theta_0 = 0$).

(a) Consider the slit to be N small slits spaced a distance $d = a/N$ apart, where N is very large. The waves emerging from the various slits do not start in phase, since the light from the source had to travel different distances to reach them. Assuming that the light arrives at the top slit ($n = 0$) with phase ϕ_0, find the phase of the light that arrives at the bottom slit ($n = N - 1$) in terms of θ_0, ϕ_0, a, and λ.

(b) Generalize your result from (a) to find the phase at the n^{th} slit ϕ_n. The n^{th} slit is at a distance $\frac{n}{N}a$ below the top slit. Write your answer in the form $\phi_n = \phi_0 + n\Delta\phi$. Express $\Delta\phi$ in terms of θ_0, ϕ_0, a, l, and N (as needed).

(c) Now use the ϕ_n to find the intensity $I(x)$ at the screen due to the N slits. [*Hint:* Follow the procedure used in section 3.2.1 of the Notes, but this time the ϕ's are not all equal.]

(d) Take the limit $N \to \infty$ to find the intensity at the screen due to the slit of width a. A correct result cannot have any n's or N's in it (why not?). Check your result for the special case $\theta_0 = 0$.

(e) Explain how the intensity pattern differs from the pattern in the "regular" setup. At what angle θ does the "central" maximum appear?

<u>ANSWERS:</u>

(a) $\phi_{N-1} = \phi_0 + ka\sin\theta_0 = \phi_0 + (2\pi a/\lambda)\sin\theta_0$.

(b) $\phi_n = \phi_0 + nk(a/N)\sin\theta_0 = \phi_0 + (2\pi na/N\lambda)\sin\theta_0$. So, $\Delta\phi = (2\pi a/N\lambda)\sin\theta_0$. (This general expression reproduces the result in (a) for very large N: $\phi_{N-1} = \phi_0 + [k(N-1)a/N]\sin\theta_0 \stackrel{N\to\infty}{\longrightarrow} \phi_0 + ka\sin\theta_0$.)

(c) The arguments of the sines in the N-slit formula have to be shifted by $\Delta\phi$:

$$I(\sin\theta) = I_0\frac{\sin^2\left[\frac{N}{2}(k\Delta R + \Delta\phi)\right]}{\sin^2\left[\frac{1}{2}(k\Delta R + \Delta\phi)\right]}\ , \qquad \Delta R \equiv (a/N)\sin\theta\ ,\ \Delta\phi = (ka/N)\sin\theta_0\ . \tag{1}$$

(d) Taking the limit in a similar way as in class gives:

$$I(\sin\theta) = I_{\max} \frac{\sin^2\left[\frac{ka}{2}(\sin\theta + \sin\theta_0)\right]}{\left[\frac{ka}{2}(\sin\theta + \sin\theta_0)\right]^2} \,. \tag{2}$$

(e) If we plot the intensity I versus θ, the central maximum will appear at $\theta = -\theta_0$.

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2. Exercise Problem 2:
Young & Freedman, Problem 38-6.

3. Exercise Problem 3:
Young & Freedman, Problem 38-16. Assume that $\theta \ll 1$ (use small-angle approximations for θ).

4. Exercise Problem 4:
Three very thin slits (widths $\ll \lambda$) are illuminated at normal incidence with laser light of wavelength λ. The slits are separated by a center-to-center distance d. The slit in the center (#2) is wider than the other two. The intensity at the screen due to the central slit alone would be $4I_0$; the intensity at the screen due to either of the other slits alone would be I_0.

(a) Write an expression for the intensity at points on the screen in terms of I_0 and $e^{ik\Delta r_{21}}$. [$k = 2\pi/\lambda$ and $\Delta r_{21} = r_2 - r_1$, where r_2 and r_1 are the distances from slits #2 and #1 to a point on the screen.]

(b) What is the maximum possible intensity at any point on the screen due to all three slits?

(c) What is the minimum possible intensity at any point on the screen due to all three slits?

(d) What is the smallest value of θ (> 0) at which the maximum possible intensity occurs? (θ is the usual angle to the screen.)

(e) What is the smallest value of θ (> 0) at which the minimum possible intensity occurs? [*Hint:* A phasor diagram may help.]

ANSWERS:

(a) $I(\Delta r_{21}) = I_0 \left|1 + 2e^{ik\Delta r_{21}} + e^{2ik\Delta r_{21}}\right|^2$.

(b) $16I_0$.

(c) 0.

(d) $\Delta r_{21} = \lambda \Longrightarrow \theta_{\max} \approx \frac{\lambda}{d}$.

(e) $\Delta r_{21} = \frac{\lambda}{2} \Longrightarrow \theta_{\min} = \frac{\lambda}{2d}$.

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5. Exercise Problem 5:
Young & Freedman, Problem 37-50. The index of refraction n of a material is the ratio of the speed of light in vacuum to the speed of light in the material. It is not hard to show that if the wavelength

of light in vacuum is λ_0, the wavelength in the material is $\lambda_n = \lambda_0/n$: since the frequencies are equal, $\lambda_n/\lambda_0 = v_n/v_0 = 1/n$.

6. Exercise Problem 6:
The graph on the last page of this Exercise Set shows the intensity (in arbitrary units) as a function of position (y) on a screen which is 1.00 m from an aperture illuminated at normal incidence by laser light of wavelength 0.628 μm. The aperture consists of some number of equally spaced, equally wide slits. Determine the number of slits N, the slit width a, and the center-to-center slit separation d.

ANSWERS: The screen intensity $I(\theta)$ for a diffraction pattern from *multiple finite slits* is given by the expression:

$$I(\theta) = \left(I_0 \frac{\sin^2(k\Delta r/2)}{(k\Delta r/2)^2}\right)\left(\frac{\sin^2(Nk\Delta R/2)}{\sin^2(k\Delta R/2)}\right) , \quad (3)$$

where $\Delta r = ka\sin\theta$ and $\Delta R = kd\sin\theta$. Looking at the picture, we therefore determine that:

- Since there are 4 secondary maxima between each two principal maxima, $N = 6$.
- The condition for a principal maximum is $k\Delta R = 2n\pi$ or for $n = 1$, $\sin\theta_1 = \frac{\lambda}{d}$. Then, $d \approx \frac{1\text{m}}{0.015\text{m}}\lambda \approx$ 42 μm.
- The height of the principal maxima is modified by the finite-slit pre-factor in (3). Comparing the heights of the $n = 0$ and $n = 1$ principal maxima on the plot, we obtain $\frac{I_{(n=1)}}{I_{(n=0)}} = \frac{\sin^2(\pi a/d)}{(\pi a/d)^2} = \frac{1.36}{1.8} = 0.76$. The ratio $\frac{a}{d} = \frac{1}{4}$ satisfies this condition reasonably well. Thus, $a \approx d/4 \approx 10.5$ μm.

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7. Exercise Problem 7:
A particle of mass m is trapped in an **infinite square well potential** with a **finite step** in the middle. The width of the well is a and the step (of height V_0) is at $x = a/2$. (See Figure 1.)

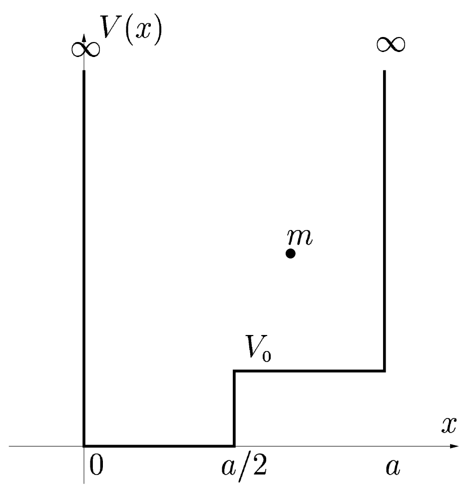

Figure 1: Infinite square well potential.

(a) ***Compute*** the (time-independent) ***wavefunctions*** of the ***two lowest-lying energy eigenstates*** of the particle.

(b) ***Compute*** the ***energies*** of the ***two lowest-lying energy eigenstates*** of the particle.

Hint: Look for solutions of the type $\psi(x) = Ae^{ikx} + Be^{-ikx}$ in each region where the potential $V(x)$ stays constant. Then "glue" the wavefunctions smoothly together, i.e., impose the requirements (BCs) that $\psi(x)$ vanish at $x = 0,\ a$, and both $\psi(x)$ and $\frac{d\psi}{dx}$ be continuous at $x = a/2$.

<u>ANSWERS:</u> We look for solutions of the type:

$$\psi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \ , \tag{4}$$
$$\psi_2(x) = A_2 e^{ik_2(x-a)} + B_2 e^{-ik_2(x-a)} \ , \tag{5}$$
$$\tag{6}$$

in the regions 1 and 2 respectively, with $k_1 = \sqrt{2mE}/\hbar$ and $k_2 = \sqrt{2m(E - V_0)}/\hbar$. Applying the boundary conditions, we get:

$$\psi_1(x = 0) = 0 \Rightarrow B_1 = -A_1 \ , \tag{7}$$
$$\psi_2(x = a) = 0 \Rightarrow B_2 = -A_2 \ , \tag{8}$$
$$\psi_1(x = a/2) = \psi_2(x = a/2) \Rightarrow A_1 \sin(k_1 a/2) = -A_2 \sin(k_2 a/2) \ , \tag{9}$$
$$\psi_1'(x = a/2) = \psi_2'(x = a/2) \Rightarrow k_1 A_1 \cos(k_1 a/2) = k_2 A_2 \cos(k_2 a/2) \ . \tag{10}$$

Rephrasing everything in terms of A, the single constant remaining undetermined after solving the first three equations such that $A_1 = (A/2i)\sin(k_2 a/2)$, and using Eqs. (7–9), we obtain:

$$\psi_1(x) = A \sin(k_2 a/2) \sin(k_1 x) \ , \tag{11}$$
$$\psi_2(x) = -A \sin(k_1 a/2) \sin(k_2 (x - a)) \ , \tag{12}$$

where the energies (related to k_1 and k_2 as shown above) have to satisfy the transcendental equation:

$$\frac{\tan(k_2 a/2)}{\tan(k_1 a/2)} = -\frac{k_2}{k_1} \ . \tag{13}$$

The two lowest energies will be the *two smallest* solutions to (13), and the two lowest states can be obtained by substituting their values into (11) and (12).

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8. Exercise Problem 8:

Now consider a particle of mass m confined in an **finite square well potential** of height V_0 and width a. (See Figure 2.)

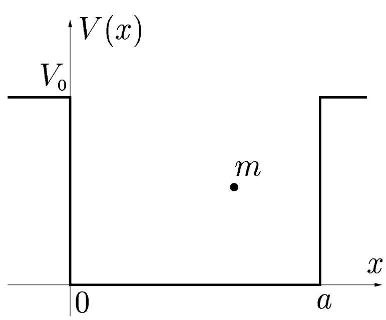


Figure 2: Finite square well potential.

(a) ***Compute*** the (time-independent) ***wavefunctions*** of the ***two lowest-energy states*** of the particle.

Hint: Use the same strategy as in the previous problem. In this case the smoothness requirements (BCs) are that both $\psi(x)$ and $\frac{d\psi}{dx}$ be continuous at $x = 0,\ a$.

(b) ***Write down the equations*** that the ***energies*** of the ***two lowest-energy states must satisfy.***

Hint: Some of the boundary conditions above will impose constraints on the wavevectors $k = \sqrt{2mE}/\hbar$ and $k' = \sqrt{2m(E - V_0)}/\hbar$, and thus on the energy E. As a result the energy can only take certain discrete values, solutions to transcendental equations (e.g., $x = \tan x$ is a *transcendental equation*). Such equations can only be solved numerically (or graphically). In this problem you are not required to solve the equations; it is enough just to write them down.

ANSWERS: We look for solutions of the type:

$$\psi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} , \tag{14}$$

$$\psi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x} , \tag{15}$$

$$\psi_3(x) = A_3 e^{ik_3(x-a)} + B_3 e^{-ik_3(x-a)} . \tag{16}$$

in the regions 1, 2, and 3 respectively, with $k_1 = k_3 = i\sqrt{2m(V_0 - E)}/\hbar = i\kappa$ and $k_2 = \sqrt{2mE}/\hbar$. Note that since we are looking for the lowest-energy bound states, we take $E < V_0$ and the wavevectors in the regions 1 and 3 are *imaginary.* The exponents in (14) and (16) are therefore *real* and in order to ensure that the wavefunction vanishes at infinity, we must set $A_1 = B_3 = 0$. (The right-moving waves that penetrate region 3 and the left-moving waves that penetrate region 1 decay exponentially; there is no right-moving waves in region 1 and no left-moving waves in region 3.)

Applying the boundary conditions at $x = 0$ and $x = a$, we get:

$$\psi_1(x=0) = \psi_2(x=0) \Rightarrow B_1 = A_2 + B_2 , \tag{17}$$

$$\psi_1'(x=0) = \psi_2'(x=0) \Rightarrow \kappa B_1 = ik_2(A_2 - B_2) , \tag{18}$$

$$\psi_1(x=a) = \psi_2(x=a) \Rightarrow A_3 = A_2 e^{ik_2 a} + B_2 e^{-ik_2 a} , \tag{19}$$

$$\psi_1'(x=0) = \psi_2'(x=0) \Rightarrow -\kappa A_3 = ik_2(A_2 e^{ik_2 a} - B_2 e^{-ik_2 a}) . \tag{20}$$

We can solve Eqs. (17–20) in terms of a single constant A (to be fixed by normalization):

$$B_1 = 2ik_2 A , \tag{21}$$

$$A_2 = A(ik_2 + \kappa) , \tag{22}$$

$$B_2 = A(ik_2 - \kappa) , \tag{23}$$

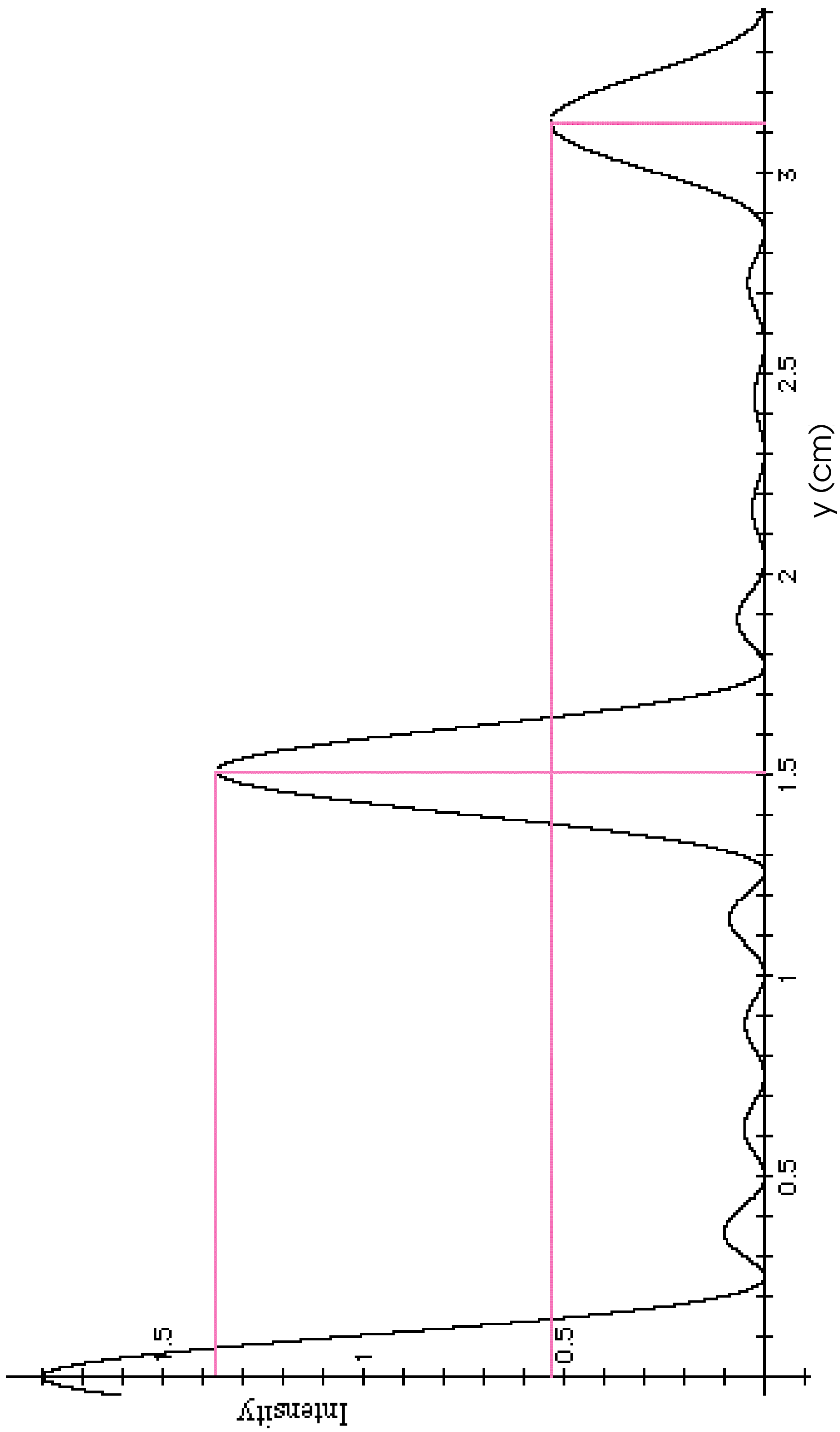
$$A_3 = 2iA(k_2 \cos(k_2 a) + \kappa \sin(k_2 a)) . \tag{24}$$

The wavefunctions are of the form (14–16) with B_1, A_2, B_2, and A_3 given by (21–24) and k_2, κ can be obtained once we know the value of the energy (recall that $k_2 = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$).

The energy can only have *discrete* values, solutions to the transcendental equation:

$$\frac{2k_2\kappa}{k_2^2 - \kappa^2} = \tan(k_2 a) . \tag{25}$$

The two lowest energies will be the *two smallest* solutions to (25), and the two lowest states can be obtained by substituting those values for the energy into (21–24) and (14–16).


Intensity
0.5
1
1.5
y (cm)
0.5
1
1.5
2
2.5
3